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Coloring maps
MATHEMATICS STAFF OF THE COLLEGE, *University of Chicago*

On teaching angle and angle measure
HOWARD F. FEHR

Trigonometric values that are algebraic numbers
KENNETH W. WEGNER

Sequence and range of high school mathematics
RICHARD F. BRUNS *and* ALEXANDER FRAZIER

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Coloring maps[1]

MATHEMATICS STAFF OF THE COLLEGE,
University of Chicago, Chicago, Illinois.

Your students will be interested in the four-color problem. "Tease" them to invent solutions. It will be fun.

Introduction

Of the many famous unsolved problems in mathematics one of the best known and most accessible to laymen has to do with the coloring of maps. In its mathematical form the problem goes back something less than a century.

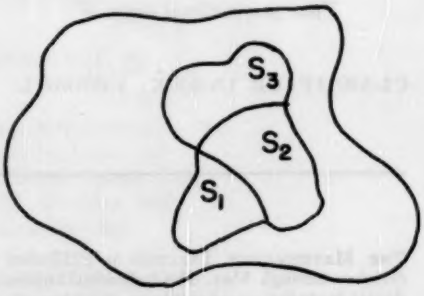
It had been noticed by British cartographers, as a matter of experience, that to color even the most varied sorts of political maps no more than four colors were ever really needed. Around 1850 this observation came to the attention of an Edinburgh student of mathematics, Francis Guthrie, who recognized in it an interesting geometrical proposition, provided it could actually be proved. He discussed the question with his brother, who communicated it to the British logician and mathematician De Morgan. From him it circulated among other British mathematicians, and the great Cayley called attention to it in a communication to the London Mathematical Society in 1878.

There have been many expository presentations of the four-color problem, as it is commonly called.[2] These provide good introductions to the problem for students and teachers alike. The purpose of this paper is different. The mode of exposition adopted here has as its principal aim to suggest ways to stimulate inventiveness in approaching mathematical problems. The technique is one of presenting an ordered progression of problems and their solutions, all having a connection with the four-color problem. To a considerable extent the exposition followed derives from a Russian book, *Mathematical Discussions*, by E. B. Dynkin and V. A. Uspenskii (Moscow-Leningrad, 1952).[3] The book originated from materials collected by one of the sections of the mathematics circle for secondary school students at the University of Moscow.

The coloring problem

A political map shows countries in different colors. Ordinarily it is not required that different countries have different colors, but only that no two border-

Figure 1



[1] This paper has been prepared by members of the Mathematics Staff of the College of the University of Chicago as part of a *Survey of recent East European literature in intermediate mathematics*. The *Survey* is supported by a grant to the Staff by the National Science Foundation.

[2] Three of these (in English) are: R. Courant and H. C. Robbins, *What Is Mathematics* (New York: Oxford University Press, 1941), pp. 246–248; D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea, 1952; translated from the German), pp. 335–340; H. Rademacher and O. Toeplitz, *The Enjoyment of Mathematics* (Princeton: Princeton University Press, 1957; translated from the German), pp. 73–82. Also (in German): H. Tietze, *Gelöste und ungelöste mathematische Probleme*, vol. II (Munich: Biederstein Verlag, 1949), pp. 48–67.

[3] This book is being translated into English in connection with the College Mathematics Staff *Survey*, and will be published soon.

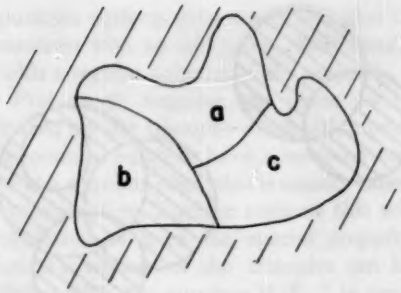


Figure 2

ing countries be colored the same. In Figure 1, countries S_1 and S_2 have a common boundary and would be differently colored, while countries S_1 and S_3 are in contact only at a point and so could be colored the same.

A coloring of a map in which no two countries with a common boundary have the same color will be called *regular*. A natural question is then: How many colors are needed to color a given map regularly? An obvious answer is: Use as many different colors as there are countries. This statement and solution of the problem miss the point. What is wanted is rather the *smallest* number of colors sufficient for the regular coloring of a given map.

How many colors are needed for a regular coloring of the map in Figure 2, representing an island in the middle of the sea? For the island itself, only three. Usually, however, the sea is colored in maps, and countries with a seacoast are colored differently from the sea. The sea in this way behaves like another country. To get around difficulties like this all maps will be considered to cover the whole plane. The answer to the question posed is that four colors are necessary.

How many colors are needed for a regular coloring of the map in Figure 3? The large country at the bottom could be given color *a* and the small country one of *b*, *c*, *d*. Then the sea could be finished with *b* or *c*. The answer is four. But not if the small country inside the large one is, as sometimes happens with political maps, regarded as a detached part of the upper

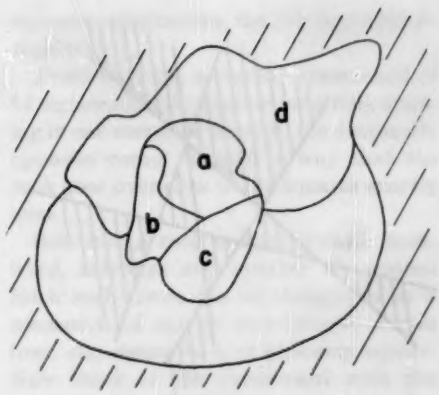


Figure 3

country which has color *a*. For then some color *e* different from *a*, *b*, *c*, *d* would have to be used for the large country. The existence of such detached political parts very much complicates the coloring of maps, and is not dealt with in the theory of the four-color problem, where each country is regarded as consisting of just one piece. With this assumption, the answer to our question is that four colors are sufficient.

Four colors are enough for a regular coloring of all these maps, and all others known. The conjectured solution of the four-color problem may be stated:

Every plane map can be regularly colored with four colors.

A disproof of the conjecture would exhibit a map that required five or more different colors for a regular coloring. Until now no such counterexample has been produced. On the other hand, it has been proved[4] that five colors suffice for the regular coloring of any map. And so things stand with a disappointing gap:

Not every map can be regularly colored with *three* colors (Figure 2).

Every map can be regularly colored with *five* colors.

The rest of this paper is concerned not with the general problem, but with maps that can be regularly colored with just *two* colors.

[4] See, for example, the exposition carried through in H. Rademacher and O. Toeplitz, *op. cit.*, pp. 74–81.

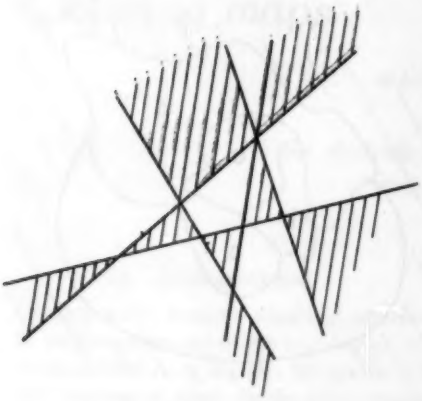


Figure 4

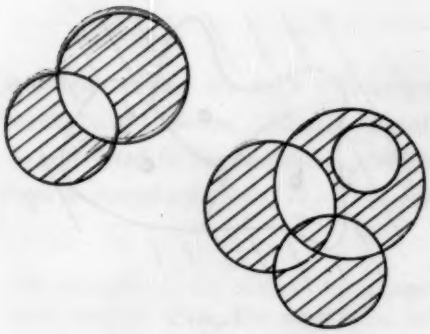


Figure 5

Some problems and their solutions

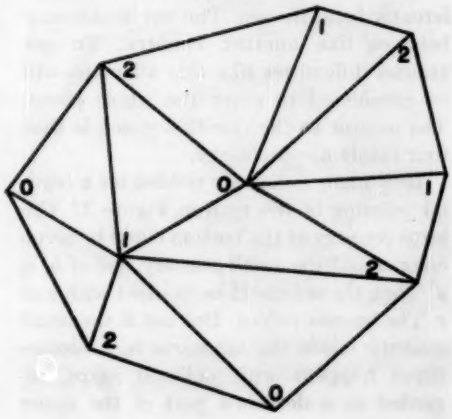
Problem 1. A map formed by drawing any number of straight lines in the plane can be regularly colored with two colors.

Solution: Let red and blue be the two colors. The proof is by mathematical induction. Red on one side and blue on the other will color a map formed by just one straight line. Now suppose it has been already proved that two colors suffice for a regular coloring of maps formed by n straight lines. Consider a map K formed by $n+1$ lines. Erase one of the lines, and there results a new map formed by n straight lines. Color this map K' regularly with red and blue, as was supposed possible. Now put back the line s which was removed. On one side of s leave all the countries of K colored as they were in K'; on the other side change red to blue and blue to red. Because K' was regularly colored, the two halves of K are each regularly colored. Now if two countries of K have a part of the line s as a common boundary, they must have resulted from the division into two of a single country of K'. Because in coloring K the colors on one side of s were left unchanged and reversed on the other, the two countries must be one red and the other blue. Therefore, K is regularly colored with two colors. Now then, since the statement about coloring is true for one line, it is true for two by what has just been shown. Being true for two, it is true for three, etc. Therefore, it is true for any number of lines.

Problem 2. A map formed by drawing any number of circles in the plane can be regularly colored with two colors.

Solution: A proof like the preceding can again be given using the method of mathematical induction. Instead a different method of proof will be exhibited. For each country, count the number of circles inside which it lies. Now if A and B are two neighboring countries having an arc of circle C as common boundary, one will be inside C and the other outside. On the other hand, A and B will be either both inside or both outside any circles other than C. Therefore the counts for A and B will differ by one. Now color red all

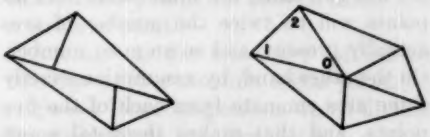
Figure 6



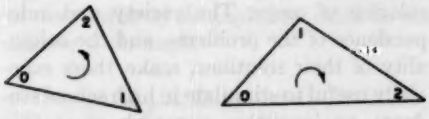
countries with an even count and blue all countries with an odd count. This clearly yields a regular coloring of the map.

Problem 3. Suppose the plane to be divided up into triangles which either have no common point, or have a common vertex or a common edge; this is usually called a triangulation. Further suppose this triangulation to have the special property that the vertices of the triangles can be labeled with the numbers 0, 1, 2 in such a way that the two ends of every edge always have different labels. A map so formed can be regularly colored with two colors.

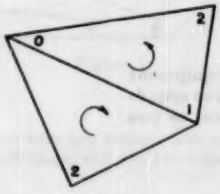
Solution: Note that maps like those below cannot be regularly colored with two colors. The first is not a triangulation; the second is a triangulation but does not have the special property.



Mark arrows on the edges going from 0 to 1, from 1 to 2, from 2 to 0. Each triangle receives in this way a definite orientation, which is the sense of the circuit from 0 to 1 to 2 to 0. The two kinds of triangles thereby distinguished are those for which the orientation is counterclockwise and those for which it is clockwise. Color all the first



kind red and all the second kind blue. Since each two neighboring triangles have



opposite orientations, the coloring must be regular.

Problem 4. On a regular chessboard of 64 squares it is impossible for a rook starting in one corner to move to the diagonally opposite corner in such a way that the rook goes over each of the squares exactly once.

Solution: A rook moves forward, backward, sideways any number of squares. Each such move can be thought of as a succession of one or more simple moves from one square to a neighboring square. Now think of the chessboard with the usual black and white pattern, a regular coloring. To cover the whole 64 squares takes the rook 63 such simple moves. An even number of simple moves always takes the rook from a square of one color to a square of the same color. Consequently the 63 simple moves must take the rook to a square of the other color from the corner square where it starts. Therefore, regardless of what path the rook may follow, this final square cannot be the diagonally opposite one, which has the same color as the corner one where the rook starts.

Problem 5. Can the whole set of 28 dominoes be arranged in a single chain in such a way that the number 6 is at one end and the number 5 at the other?

Solution: Each domino consists of two halves, each half carrying one of the numbers 0, 1, 2, 3, 4, 5, 6. There are as many dominoes in the set of 28 as there are pairs of these numbers. Each number occurs once paired with each one of the other six numbers, and once paired with itself in a double. The total number of occurrences of each one of the seven numbers is therefore eight, an even number. Now in a chain, two dominoes which are joined always show the same number on the two touching halves. It follows that in the interior of the chain each of the seven numbers must appear an even number of times. Because in the whole chain each number must appear eight times, since all the dominoes are used, a number cannot appear at one end only. So if a 6 is at one end of a

complete chain of dominoes, then 6 must be at the other end too.

Problem 6. Every human being who has ever lived has shaken hands some perfectly definite number of times, not excluding zero. The number of people who have shaken hands an odd number of times must be even.[5]

Solution: Let the number of human beings who ever lived be N, and the total number of handshakes exchanged be m. Number the people from 1 to N, and let n_k be the number of times the person numbered k has shaken hands, not excluding zero. Now add up the number of times all the people have shaken hands. This counts twice each handshake exchanged, so that the result is

$$n_1+n_2+\cdots+n_N=2m.$$

But if, as here, the sum of a group of numbers is even, then the number of odd summands occurring must also be even.

Problem 7. At the opening session of a meeting at which 225 people were present, friends shook hands with one another. When these greetings were concluded, there must have been at least one person present who had shaken hands an even number of times.

Solution: The number 225 is odd. See Problem 6.

Problem 8. In Figure 7, above, six points are given; each of them is shown connected with three of the remaining five points. If instead of six only five points are given, it is impossible to draw arcs connecting each point with exactly three of the remaining points.

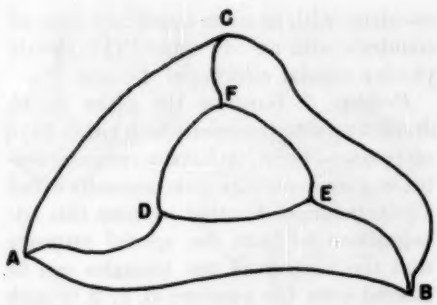


Figure 7

Solution: Suppose the arrangement with five points is possible. Count the arcs emanating from each point. Since each arc has two ends, the total count over all points will be twice the number of arcs actually present, and so an even number. On the other hand, by assumption exactly three arcs emanate from each of the five points, and that makes the total count over all points 15, which is an odd number. The supposed arrangement with five points is therefore impossible.

Conclusion

The foregoing eight problems, with solutions, are representative of those in the Russian book, which relate to two colors. All connect in some way with the regular coloring of maps. The variety and independence of the problems, and the originality of their solutions, make them especially useful to stimulate in high school students an inventive approach to mathematical problems.

[5] This is one of a number of problems mentioned by B. V. Gnedenko as being presented in preparation for the eleventh Moscow Olympiad, which is a mathematical contest. See his paper *Mathematical education in the U.S.S.R.*, American Mathematical Monthly, vol. 64 (1957), pp. 389–408.

A "quiet room" used in reducing equipment noise of a U.S. aircraft manufacturer is so soundproof that you can hear your heartbeat as you walk through it.—*Planes.*

On teaching angle and angle measure

HOWARD F. FEHR, *Teachers College, Columbia University, New York City.*

An algebraic approach to some problems untouched by classical Euclidean geometry.

PLANE ANGLE

WE ASSUME POINT, straight line, and plane as undefined geometric entities and accept the usual drawings as illustrations of these entities. A *ray* or *half-line* consists of a point on a line and all the points of the line to the one side of this point. A ray has an initial point but no terminal point. A closed line segment has both an initial and a terminal point.

An *angle* is the line-figure formed by two rays having the same initial point. An angle is a line configuration which divides the entire plane into two sectorial regions of infinite extent.[1] For either of these regions, if two points are selected in the region, they can always be connected by a broken line segment (see *ZTVW* in Figure 1) which lies entirely in the region and does not intersect the angle. In the figure *X* and *Y* are points in one region, while *Z* and *W* are points in the other region. One of these regions is specified as the *interior of the angle*, the other as the *exterior of the angle*.

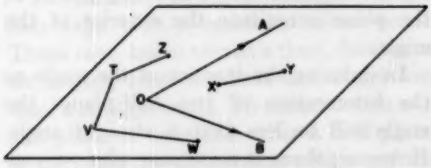


Figure 1

[1] If the two rays are distinct but collinear, they form a straight angle, and the two regions are then half-planes.

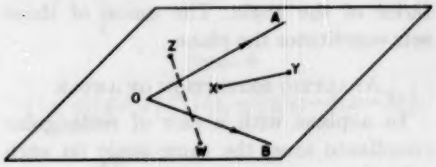


Figure 2

The question arises as to how we can specify which region is the interior and which is the exterior of the angle. If we limit the concept of angle to a configuration less than a straight angle (2 collinear rays), as Hilbert and Veblen do,[2] the problem is easy. One of the two regions is such that for any two points selected in it, the line segment joining these points never intersects the angle. This region is then called the interior of the angle. In the other region, however, it is easy to find two points such that the line segment joining them does intersect the angle. This region is designated as the exterior of the angle.

In trigonometry it is desirable to consider an angle as being greater than a straight angle. For this purpose we modify our definition of interior and exterior. The angle still consists of two rays having the same initial point. The angle still divides the entire plane into two regions as defined in the previous paragraph. The region in which a line segment joining any two points does not intersect the angle can be

[2] Hilbert, David, *The Foundations of Geometry*, translated by E. J. Townsend (Open Court Publishing Co., La Salle, Illinois, 1938). Veblen, T., *Foundations of Geometry. Monographs on Modern Topics in Mathematics* (Dover Publications, reprint 1955).

optionally specified as the interior region or exterior region. If it is designated as the exterior region the angle is referred to as *reflex*. On a drawing we can use a small circular arc to designate the interior of the angle.

Thus, all the points of a plane can be separated into three distinct sets of points, namely (1) an angle or the set of points on two rays with the same initial point, (2) the set of points of the interior of the angle, and (3) the set of points of the exterior of the angle. The union of these sets constitutes the plane.

Analytic definition of angle

In a plane with a pair of rectangular coordinate axes, the same scale on each axis, the system

$$\begin{Bmatrix} y=ax+b \\ y=cx+d \end{Bmatrix}$$

defines a pair of lines. If $a \neq c$, these lines intersect. From the point of intersection there are four rays which taken two at a time define six angles, two of which are straight angles. To give an analytic definition of an angle, we shall first limit ourselves to angles less than a straight angle and use specific equations for our lines.

Consider the lines

$$\begin{Bmatrix} y=x \\ y=-\frac{1}{2}(x-3) \end{Bmatrix}$$

which intersect at (1, 1). We shall designate the rays of these lines by limiting the domain of x, that is, permitting x to take on real values only in a certain interval. Thus, the union[3] of the sets of points defined by

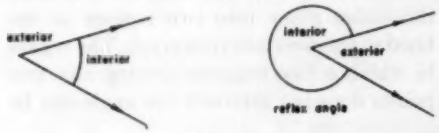


Figure 3

[3] Union is defined as the set of all points belonging to either one or the other of the two rays. The symbol for union is $\cup$.

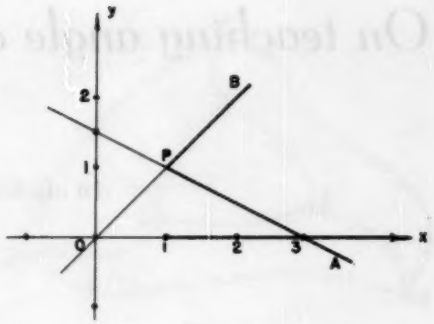


Figure 4

$$\begin{Bmatrix} y=x, & x \geqq 1 \\ y=-\frac{1}{2}(x-3), & x \geqq 1 \end{Bmatrix}$$

defines an angle. The angle is the set of points of two rays each having the same initial point P.

The angle has an interior and an exterior region each of which must be designated analytically. For this purpose we use the concept of half-plane. A line divides a plane into two regions either of which is called a half-plane, and for either the line is a boundary. Thus, a half-plane is to a plane as a ray is to a line. Two distinct lines have four distinct half-planes. The intersection[4] of two half-planes, one from each line, determines the *interior of the angle*. The figures show the four possible intersections of either half-plane of one line with either half-plane of the other line. Thus, the interior of an angle is designated as the intersection of two properly chosen half-planes. The angle then consists of the set of points on the rays bounding the interior. The remaining points of the plane constitute the exterior of the angle.

In defining the interior of the angle as the intersection of two half-planes the angle will be less than a straight angle. However, there is no reason why we cannot designate the intersection as the exterior of the reflex angle and the other

[4] Intersection is defined as the set of all points belonging to both of two given sets of points. The symbol for intersection is $\cap$.

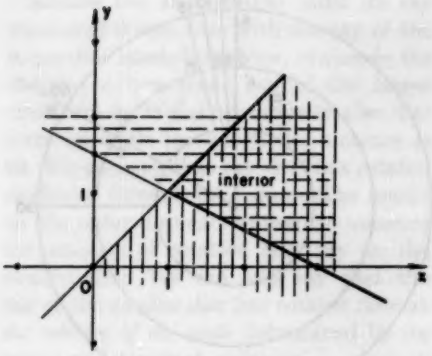


Figure 5

$\{(x, y) | y < x\} \cap \{(x, y) | y > -\frac{1}{2}(x-3)\}$[5]

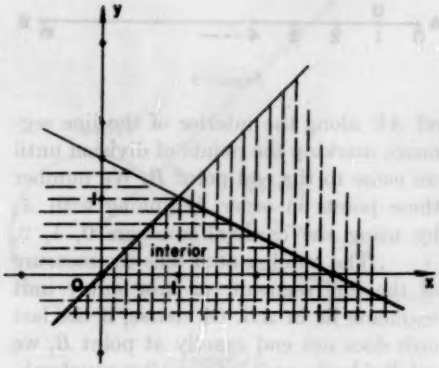


Figure 6

$\{(x, y) | y < x\} \cap \{(x, y) | y < -\frac{1}{2}(x-3)\}$

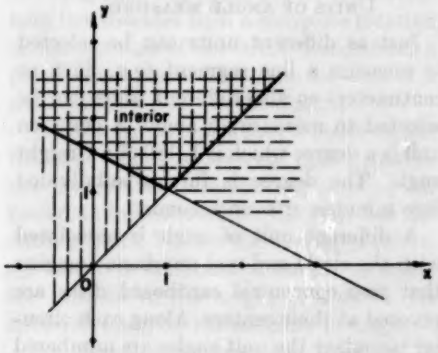


Figure 7

$\{(x, y) | y > x\} \cap \{(x, y) | y > -\frac{1}{2}(x-3)\}$

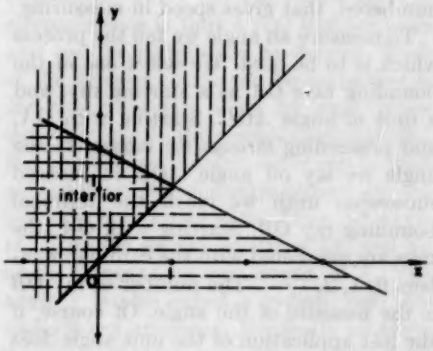


Figure 8

$\{(x, y) | y > x\} \cap \{(x, y) | y < -\frac{1}{2}(x-3)\}$

bounded region as the interior of this angle, especially if we wish to concern ourselves with angles greater than a straight angle.

In summary, two intersecting lines form four rays with a common initial point. These rays, taken two at a time, determine six angles, four of which are angles less than a straight angle. The intersection of a half-plane of one line with a half-plane of the other line determines a region of the plane bounded by two rays. This region is usually designated the interior of the angle. (But it may be designated the exterior of the angle.) The complement,[6] with respect to the plane, of the union of the angle and its interior is the exterior of the angle.

Measurement of angles

In measuring a line segment, we select one bounding point A as a starting point and a unit line segment AU. Then we lay

[5] The statement $\{(x, y) | y < x\} \cap \{(x, y) | y > -\frac{1}{2}(x-3)\}$ means the intersection ($\cap$) of the set of points (x, y) for which $y < x$ and the set of points (x, y) for which $y > -\frac{1}{2}(x-3)$.

[6] The complement of a set of points A with respect to a universe of which A is a subset, is the set of points in the universe which do not belong to A. The symbol for complement is $\tilde{A}$. Thus, $\tilde{A} = U - A$ or $A \cup \tilde{A} = U$.

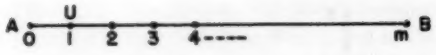
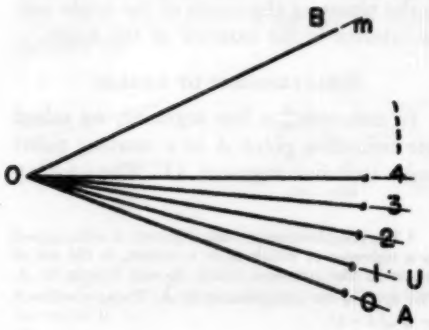


Figure 9

off AU along the interior of the line segment, marking the points of division until we come to the end point B. We number these points in order, beginning with A, by using the cardinal numbers 0, 1, 2, · · · . The number at B tells the measure of the line segment, or how many unit segments lie in AB. Of course, if the last unit does not end exactly at point B, we subdivide the unit into smaller equal subunits and use these subunits to obtain a more precise measure of AB. The ruler is an instrument, already subdivided and numbered, that gives speed in measuring.

To measure an angle we tell the process which is to be used. We select one of the bounding rays OA as a starting ray, and a unit of angle AOU. Starting with OA, and proceeding *through the interior* of the angle we lay off angle AOU in ordered succession until we reach the terminal bounding ray OB. Starting with OA, the rays are numbered with the cardinal numbers 0, 1, 2, · · · . The number of ray OB is the measure of the angle. Of course, if the last application of the unit angle does not end on OB, we can subdivide the unit angle into equal subdivisions to obtain a more precise measure of the angle. The protractor is an instrument already subdivided and numbered, that permits speed in measuring an angle.

Figure 10



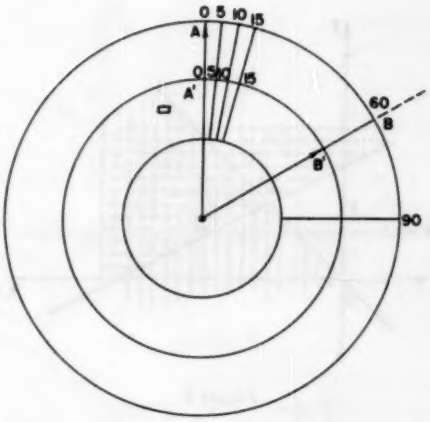


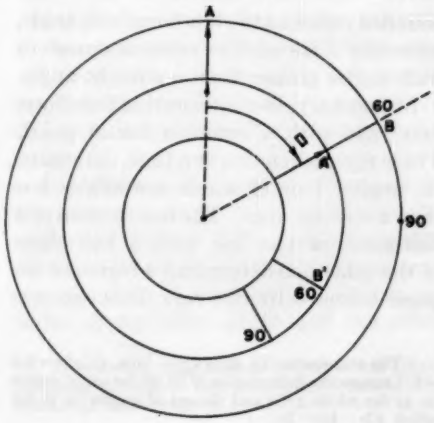
Figure 11

Units of angle measure

Just as different units can be selected to measure a line segment (e.g., inch or centimeter) so also different units can be selected to measure an angle. A common unit is a degree which is 1/180 of a straight angle. The degree is further subdivided into minutes or even seconds.

A different unit of angle is associated with the circle and real numbers. Imagine that two concentric cardboard discs are pivoted at their centers. Along each circular boundary the unit angles are numbered clockwise, and the rays of both discs with the same numbers coincide.

Figure 12



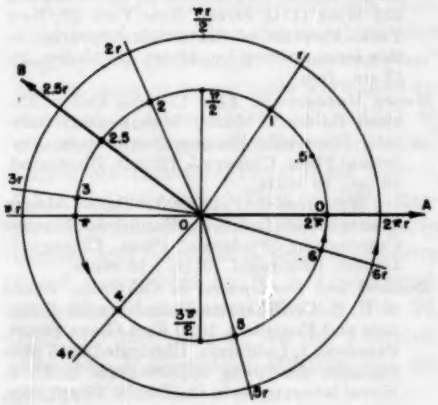
Rotate the smaller disc until its ray numbered 0 coincides with the ray of the larger disc labeled 60. Now, obviously the initial (or 0-marked) ray of the larger circle and the initial ray of the smaller disc form an angle for which the measure is 60. We also say that ray OA' has rotated clockwise through 60 degrees. The marks on the outer disc can be used to measure the amount of rotation of a ray on the smaller disc. We can also say that the ray of the smaller disc has rotated *through the interior of an angle* determined by its initial and terminal positions.

If a ray rotates in a given direction (clockwise or counterclockwise) until it again reaches its initial position, the ray has swept out the entire plane. We cannot help but consider such a complete rotation as the measure of the entire plane about a point in the plane. Thus, the measure of a plane about a point in the plane is 360°.

In drawing a circle about a point we get the same feeling of coverage (going around the plane). Whereas the angles involved bound an infinite surface, the length of the circle is finite. We use this feeling of coverage and this finiteness of length to relate the measure of the length of a circle (its circumference) to the measure of angle.

For convenience we use a circle of radius one and center at the vertex of the angle. The circumference of the circle is

Figure 13



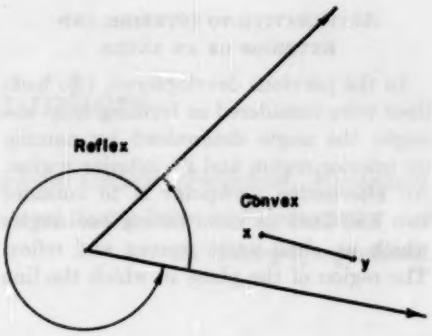


Figure 14

divided into unit lengths, the division points numbered, and the closing or circumference point has the value 2π. The circumference is thus a real number scale and can be subdivided as desired. Each of the division marks can be connected to the vertex of the angle by rays. Since equal arc lengths have equal central angles, the numbers on the unit circle can be used as measures of the angles formed by the initial and terminal rays. Thus, an angle of $2\frac{1}{2}$ means an arc length of $2\frac{1}{2}$, and an angle 1 means an arc length of 1. The angle 1 is also called a radian, but this name is not necessary. Evidently an angle 2π is one that measures the entire plane about a point. Thus, the real numbers in the interval $(0 \leqq r \leqq 2\pi)$ are sufficient to measure any portion of a plane that is the interior of an angle.

If a circle with other than unity as radius is used, we number the circle by using the radius as a unit of measure. Since the circumference of this circle is $2\pi r$, we note that 2π again suffices to measure the interior of any angle in a plane. It is possible to use other units than the degree and the real numbers to measure an angle, but these are most common and suffice for most practical applications. However, the close association of the real numbers on a circle with the real-axis makes the real number measure the more important unit of angle so far as mathematics is concerned.

Alternative to interior and exterior of an angle

In the previous development, two half-lines were considered as forming only one angle, the angle determined by naming its interior region and its exterior region. An alternative viewpoint is to consider two half-lines as determining *two* angles which we shall name convex and reflex. The region of the plane in which the line segment connecting any two points of the region has every point of the line segment belonging to the region forms a convex set of points. In this case we refer to the angle formed by the half-lines as a convex angle. If, however, we are concerned with the other region of the plane we refer to the angle as *reflex*. The sum of the measures of the two angles thus considered will be 360°.

What's new?

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Modern Mathematics and Its Place in the Secondary School. Commission on Mathematics, 425 West 117th Street, New York 27, New York. Preprint of the article appearing in this issue written by Albert E. Meder, Jr. 13 pp., free.

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Money Management, Your Shelter Dollar. Money Management Institute, Household Finance Corporation, Prudential Plaza, Chicago 1, Illinois. Illustrated. 36 pp., 10 cents.

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Trigonometric values that are algebraic numbers

KENNETH W. WEGNER, *Carleton College, Northfield, Minnesota.*

Here is a neat tie-up between trigonometry and the theory of equations.

EVERY HIGH SCHOOL or college trigonometry student becomes familiar with trigonometric tables giving natural values. In a course in algebra, many of these same students learn to solve linear and quadratic equations and to approximate the real roots of equations of degree three and higher. Few of these students become aware of the close relationship between these two branches of their studies. Actually, *every* number in a natural functions trigonometric table in which the angle is measured in degrees and fractions of degrees is a rounded-off value of a number that is algebraic. (An algebraic number is a root of an equation of the form

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$$

with rational coefficients.) For a proof of this see E. Swift, *American Mathematical Monthly*, Vol. 29, 1922, p. 404.

Also some trigonometry students probably wonder whether the numbers they look up in a table might "come out even" if a 10-place or larger table were used instead of a 4- or 5-place table; that is, whether the trigonometric values represented are rational. (Those other than 0, 0.5, and 1, of course.) That *all* such trigonometric values in tables with the angle in degrees and fractions of degrees are *irrational* was proved by R. S. Underwood, *American Mathematical Monthly*, Vol. 29, 1922, p. 346, and J. M. H. Olmsted, *American Mathematical Monthly*, Vol. 52, 1945, p. 507.

The purpose of this article is to exibit some of the equations having trigonometric values as roots, and to urge teachers to use them as illustrations of the above facts. Some of the equations of higher degree might well serve as enrichment material for independent work by the better students.

The illustrations given here will be limited to cosine (or sine) values. Let us first consider equations whose roots are among the values cos 0° (sin 90°), cos 1° (sin 89°), cos 2°, cos 3°, ... , cos 89°. We shall see that every one of these 90 values is a root of an irreducible equation of degree 1, 2, 3, 4, 6, 8, 12, 16, 24, or 48.

The only rational values among the 90 are 1 and 0.5, giving equations:

(1) $x-1=0$, root cos 0°, and

(2) $2x-1=0$, root cos 60°.

Two quadratics come to mind at once:

(3) $4x^2-3=0$, roots $\pm\cos 30°$, and

(4) $2x^2-1=0$, roots $\pm\cos 45°$.

The only other quadratic with rational coefficients whose roots are among those sought is:

(5) $4x^2-2x-1=0$, roots cos 36°, and $-\cos 72°$, or $(1\pm\sqrt{5})/4$.

The student can verify this by getting these roots in decimal form and comparing them with the table values of cos 36° and cos 72°. Or he can work out the identity

$\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t$, substitute $t = 36°$ and $t = 108°$, thus showing that $\cos 36°$ and $\cos 108°(-\cos 72°)$ are roots of $16x^5 - 20x^3 + 5x + 1 = 0$, take out the rational root -1, and discover that he has left $(4x^2 - 2x - 1)^2 = 0$. Another procedure to show that $\cos 36°$ is a root: Using $\cos 3t = 4\cos^3 t - 3\cos t$, obtain

$$4\cos^3 36° - 3\cos 36° = \cos 3(36°)$$
$$= \cos 108° = -\cos 72° = -\cos 2(36°)$$
$$= -(2\cos^2 36° - 1) = -2\cos^2 36° + 1.$$

So $\cos 36°$ is a root of $4x^3 + 2x^2 - 3x - 1 = 0$, which is $(x+1)(4x^2 - 2x - 1) = 0$. Use a similar procedure to show that $-\cos 72°$ is the other root of the quadratic.

Only one cubic is on the list:

(6) $8x^3 - 6x - 1 = 0$, roots $\cos 20°$, $-\cos 40°$, and $-\cos 80°$.

This is easily obtained by substituting $t = 20°$, $100°$, and $140°$ in

$$\cos 3t = 4\cos^3 t - 3\cos t.$$

The student can verify that these roots are irrational by noting that the only possible rational roots are ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, and $\pm\frac{1}{8}$, none of which are actually roots.

Another approach is for the teacher to give the student the equation and ask for the roots. When the student comes with the approximations 0.940, -0.766, and -0.174, ask him to hunt for these values in a sine or cosine table. He will "discover" that they are rounded values of $\cos 20°$, $-\cos 40°$, and $-\cos 80°$. If the student has had enough calculus to use it in graphing, he will find that the bend points of his cubic curve occur at rational values of $x(\pm\frac{1}{2})$.

As another variation (which also applies to several equations yet to be considered), make the substitution $z = 1/x$, obtaining $z^3 + 6z^2 - 8 = 0$ with roots $\sec 20°$, $-\sec 40°$, and $-\sec 80°$. The student will see the advantage of working with an equation with leading coefficient unity and with roots more easily isolated. Also it will give the teacher a chance to expose him to the term "algebraic integer." (An algebraic integer is a number that is a root of an equation of the form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

with integral coefficients.)

The gifted student can go further by looking up the algebraic solution of a cubic equation. He will find that equation (6) comes under the so-called irreducible case (notice it is a reduced cubic), and will be delighted to be led directly to roots $\cos 20°$, $\cos 140° = -\cos 40°$, and $\cos 260° = -\cos 80°$. It is also of interest to check the formula concerning the sum of the roots: Since the coefficient of x^2 is zero, $\cos 20° + (-\cos 40°) + (-\cos 80°) = 0$, or $\cos 20° = \cos 40° + \cos 80°$, which checks in the trigonometric formula for the sum of two cosines.

Three quartic equations come into the picture. Two of them are of quadratic type and so can be solved directly:

(7) $16x^4 - 16x^2 + 1 = 0$, roots $\pm\cos 15°$, and $\pm\cos 75°$, or $\pm\frac{1}{4}\sqrt{2 \pm \sqrt{3}}$, and

(8) $16x^4 - 20x^2 + 5 = 0$, roots $\pm\cos 18°$, and $\pm\cos 54°$, or $\pm\frac{1}{4}\sqrt{10 \pm 2\sqrt{5}}$.

Again these can be obtained from identities, the first by substituting $t = 15°$, $75°$, $105°$, and $165°$ in

$$\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$$

and the second by substituting $t = 18°$, $54°$, $90°$, $126°$, and $162°$ in

$$\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t.$$

The student might also be asked to form these equations given that the above radical values are to be roots, justifying that these values are $\pm\cos 15°$, $\pm\cos 75°$, $\pm\cos 18°$, and $\pm\cos 54°$ by using

$$\cos t/2 = \pm\sqrt{\frac{1+\cos t}{2}}$$

with $t = 30°$, $150°$, $36°$, and $108°$. (For cosines of the latter two see the discussion of quadratics above.)

The other quartic is of more interest:

(9) $16x^4+8x^3-16x^2-8x+1=0$,

roots $\cos 12°$, $-\cos 24°$,

$-\cos 48°$, and $\cos 84°$.

It is obtained by substituting $t=12°$, $60°$, $84°$, $132°$, and $156°$ in the above expression for $\cos 5t$, getting $32x^5-40x^3+10x-1=0$, and then taking out the factor $2x-1$. As usual, the student can verify by using the rational root theorem that all four roots are irrational. Of course, the assignment of this equation need not require all four roots. A student might be asked to calculate to three decimal places the root near $x=1$. When he returns to check his value $x=0.978$, ask him to do it by looking up $\cos 12°$. If one tries Ferrari's solution on this quartic he will find that the resolvent cubic equation has no rational root.

There are no 5th or 7th degree equations with rational coefficients having as a root the cosine of an integral number of degrees, and there is only one of degree 6:

(10) $64x^6-96x^4+36x^2-3=0$,

roots $\pm\cos 10°$, $\pm\cos 50°$, $\pm\cos 70°$.

Obtain this by letting t be $10°$, $50°$, $70°$, $110°$, $130°$, and $170°$ in

$$\cos 6t=32\cos^6 t-48\cos^4 t+18\cos^2 t-1.$$

Again verify that all of the roots are irrational. In plotting

$y=64x^6-96x^4+36x^2-3$ to

approximate the roots, the bend points will be found at $x=0$, $\pm\frac{1}{2}$, $\pm\sqrt{3}/2$, and it will be quickly noted that all the roots lie between -1 and 1. Of course, because of the symmetry only the right side of the curve need be drawn.

The better student might be asked to find these roots another way—using the transformation $z=x^2$, obtaining the roots of $64z^3-96z^2+36z-3=0$, and taking their square roots. He could then check his results in the trigonometric table.

The next two equations are of degree 8:

(11) $256x^8-448x^6+224x^4-32x^2+1=0$,

roots $\pm\cos 6°$, $\pm\cos 42°$,

$\pm\cos 66°$, and $\pm\cos 78°$, and

(12) $256x^8-512x^6+304x^4-48x^2+1=0$,

roots $\pm\cos 9°$, $\pm\cos 27°$,

$\pm\cos 63°$, $\pm\cos 81°$.

These are obtained from substitutions in the identity for $\cos 10t$, reducing the resulting 10th degree equation by taking out $4x^2-3$ in the first case and $2x^2-1$ in the second case. These are similar to the 6th degree equation above except that, since the final coefficient is unity, the reciprocal transformation is again convenient and leads to secant values which are algebraic integers. Notice that for the second of these equations the roots can be obtained by substituting for $\cos t$ in

$$\cos t/2=\pm\sqrt{\frac{1+\cos t}{2}}$$

the four values $\pm\frac{1}{4}\sqrt{10\pm2\sqrt{5}}$ given above for $\pm\cos 18°$ and $\pm\cos 54°$. Hence, again the equation can be formed by building it up from the radical values of its eight roots.

There is a 12th degree equation which is perhaps the most interesting of all since it is the only one of degree 4 or higher of the group being discussed for which the roots do not occur in pairs of negatives. To obtain it, substitute in the identity for $\cos 15t$ fifteen values of t for which $\cos 15t=\frac{1}{2}$ ($4°$, $20°$, etc.). An equation of degree 15 is obtained that includes among its roots $\cos 20°$, $-\cos 40°$, and $-\cos 80°$ which are the roots of equation (6) above. Taking out these three roots, we obtain:

(13) $4096x^{12}-12288x^{10}+512x^9+13824x^8$

$-1152x^7-7168x^6+864x^5$

$+1680x^4-248x^3-144x^2+24x+1=0$,

roots are the cosines of

$4°$, $28°$, $44°$, $52°$, $68°$, $76°$,

and the negatives of the cosines of

$8°$, $16°$, $32°$, $56°$, $64°$, and $88°$.

It would hardly be in order to assign the procuring of even one root of this equation to many decimal places unless a calculating machine were available. Suppose one were to graph the left side of this equation. He would soon note that it has the value one for $x=0$, $\pm\frac{1}{2}$, and ± 1, whereas it is greater than 7,000,000 at both $x=2$ and $x=-2$. The derivative gives a slope of 432 at the point (1, 1) and a slope of -144 at the point $(-1, 1)$. One soon suspects that all the real roots are between 1 and -1. It might be well at this point to tell the student what the roots are. He can then plot them from a trigonometric table and get the complete picture of the situation. Note that the coefficient of x is zero, from which it follows that

$$\begin{aligned}\cos 4^\circ+\cos 28^\circ+\cos 44^\circ&\\ +\cos 52^\circ+\cos 68^\circ+\cos 76^\circ&\\ =\cos 8^\circ+\cos 16^\circ+\cos 32^\circ&\\ +\cos 56^\circ+\cos 64^\circ+\cos 88^\circ,&\end{aligned}$$

which can be verified from tables.

The identity for $\cos 12t$ yields a less interesting 12th degree equation with roots $\pm\cos 5^\circ$, $\pm\cos 25^\circ$, $\pm\cos 35^\circ$, $\pm\cos 55^\circ$, $\pm\cos 65^\circ$, and $\pm\cos 85^\circ$.

The remaining three equations of the group are also too cumbersome to use as illustrations and will only be mentioned. There is a 16th degree equation having for its roots the cosines and their negatives of 3°, 21°, 33°, 39°, 51°, 57°, 69°, and 87°. There is a 24th degree equation having for its roots the cosines and their negatives of 2°, 14°, 22°, 26°, 34°, 38°, 46°, 58°, 62°, 74°, 82°, and 86°. And, finally, a 48th degree equation has roots which are cosines and negative cosines of 1°, 7°, 11°, 13°, 17°, 19°, 23°, 29° 31° 37°, 41°, 43°, 47°, 49°, 53°, 59°, 61°, 67°, 71°, 73°, 77°, 79°, 83°, and 89°.

If you check you will find that every one of the numbers cos 0°, cos 1°, cos 2°, cos 3°, ... cos 89° occurs once and only once as a root in the 17 equations listed or mentioned above. That this is correct and that there are no other equations in the group can be verified by theorems on degrees of algebraic numbers in the *American Mathematical Monthly*. (See R. S. Underwood and C. F. Gummer, Vol. 29, 1922, p. 255, and D. H. Lehmer, Vol. 40, 1933, p. 165.) The theory of prime numbers is involved, as you may have suspected when you saw the list of degrees related to the 48th degree equation, for all those numbers are relatively prime to 90.

So far we have considered only equations whose roots are cosines of integral numbers of degrees. One might expect that, if the angles were allowed to be any rational number of degrees (roots of form $\cos(m/n)^\circ$, m and n integers), many additional equations with rational coefficients could be found. This is not the case. There are no additional quadratics. There is one additional cubic:

$$8x^3+4x^2-4x-1=0,$$

roots $\cos(180/7)^\circ$, $-\cos(360/7)^\circ$, $\cos(540/7)^\circ$. (14)

There is also but one additional quartic:

$$8x^4-8x^2+1=0,$$

roots $\pm\cos 22.5^\circ$ and $\pm\cos 67.5^\circ$, or $\pm\sqrt{2\pm\sqrt{2}}$. (15)

There were no 5th degree equations in the group above, but now we have one:

$$32x^5+16x^4-32x^3-12x^2+6x+1=0,$$

roots $\cos(360/11)^\circ$, $\cos(720/11)^\circ$, $-\cos(900/11)^\circ$, $-\cos(540/11)^\circ$, $-\cos(180/11)^\circ$. (16)

There are three additional 6th degree equations:

$$64x^6+32x^5-96x^4-48x^3+32x^2+16x+1=0,$$

roots $\cos(60/7)^\circ$, $-\cos(120/7)^\circ$, $-\cos(240/7)^\circ$, $\cos(300/7)^\circ$, $-\cos(480/7)^\circ$, $-\cos(600/7)^\circ$ (17)

$$64x^6-112x^4+56x^2-7=0,$$

roots $\pm\cos(90/7)^\circ$, $\pm\cos(270/7)^\circ$, $\pm\cos(450/7)^\circ$ (18)

(19) $64x^6+32x^5-80x^4-32x^3+24x^2+6x-1=0$,

roots $\cos(360/13)°$, $\cos(720/13)°$, $\cos(1080/13)°$, $-\cos(900/13)°$, $-\cos(540/13)°$, $-\cos(180/13)°$.

No 7th degree equations are obtained.

To summarize—there are only 16 equations with rational coefficients and of degree 7 or less having roots of the form $\cos(m/n)°$, where m and n are relatively prime integers; ten of them occur when $n=1$, one when $n=2$, three when $n=7$, one when $n=11$, and one when $n=13$. Of the 16 equations, two are linear, three are quadratics, two are cubics, four are quartics, one is a quintic, and four are of the 6th degree.

Every teacher of trigonometry, college algebra, and theory of equations should have ready access to these 16 equations for use in the classroom and for special assignments.

One must not lose sight of the fact that the trigonometric functions are transcendental rather than algebraic. That is to say, not all of their values satisfy an equation of the form

$$x^n+a_1x^{n-1}+a_2x^{n-2}\cdots+a_n=0$$

with rational coefficients. It should be pointed out to the student that the cosine of 3 radians (or any trigonometric function of a rational number of radians) is transcendental since it does not satisfy such an equation. This was proved by F. Klein in 1897.

Have you read?

BOAS, R. P., JR. "If This Be Treason," *Mathematical Monthly*, April 1957, pp. 247–249.

This little article is a must not only for mathematics teachers but also curriculum directors. The author carries the curriculum in mathematics through its circles, from square roots by a series of approximations through the algorithm, and back to a series of approximations. He gives several other examples and points out how we are suckers for such changes. He shows why mathematics is one of the few courses where every instructor must teach the preceding course, and why practice is so essential in the study of mathematics. You will be interested in his conclusion and the implications for the old and new. Read this by all means.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

GARDNER, MARTIN. "Mathematical Games," *Scientific American*, January, 1957, pp. 138–142.

The magic square has provided countless hours of mathematical thought throughout the ages and squares of all types have been constructed. But here is one that appears to be different from the others. It is absurdly simple, being only an addition table arranged in a particular way with two sets of numbers. By picking a number in a cell and taking out the row and column in which it appears and repeating this process until all cells are picked or taken out, you can then predict the sum of the numbers picked. This ability to predict comes from the sets of numbers with which one sets up the magic square originally. This is a puzzle your students will like because it is simple to construct while at the same time it is difficult to analyze the method of construction. Why not read it, construct several, and let your students figure out the procedure used. They will like it.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

JONES, PHILLIP S. "Continuity in a Mathematics Program," *The University of Michigan School of Education Bulletin*, March 1957, pp. 86–90.

No one disagrees with either the idea or need for continuity in the study of any subject. Mr. Jones argues the thesis that the acceptance of the meaning approach and the learning as a continuous process requires recurring themes at all levels of instruction. This in turn implies teaching the structure of mathematics, which really means a classification of the unifying concepts from kindergarten through college.

Would you agree that number, set operation approximation, probability, or symbolism shows central themes? How much can the kindergarten student do in statistics? Is he capable of mathematical thoughts? Is abstraction possible, even if desirable, at this level? How refined a vocabulary should we use? Is proof beneficial or essential? I think you will want to read this article in the light of present developments.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Sequence and range of high school mathematics

RICHARD F. BRUNS, *Houston Public Schools, Houston, Texas, and*
ALEXANDER FRAZIER, *Phoenix Union High School and Junior College System, Phoenix, Arizona.*

Here's a look at course offerings in large cities.

IN EVALUATING THE ADEQUACY of the high school program, continuous consideration must be given to the proper placement and sequence of subject offerings and to their range.

Such consideration is particularly pressing in a field like mathematics, in which the skills and understandings acquired at one level may become the specific tools of learning at another or may be prerequisite to success in another subject area. Not only must there be a justifiable progression established for mathematics courses as such, but their coordination with other subject fields must be thoughtfully planned. Moreover, in these days, the great awareness of the importance of insuring that there will be well-prepared students for careers that involve a foundation in mathematics has led to widespread concern for reviewing the range and content of present offerings in the high schools.

NATURE OF SURVEY

To ascertain how large city school systems[1] in the United States have met this problem and to discover what sequences they have developed in organizing high school mathematics programs, the writers made a survey of the Houston school district and of thirty-five other large school districts throughout the nation and herein report their preliminary findings. Questionnaires used in this survey sought the following information:

1. Do you follow one year of algebra, ordinarily offered in the ninth grade, with a year of plane geometry before proceeding to the other mathematics courses, or do you have some other sequence?
2. What are the various advanced courses offered your students who are following a college preparatory curriculum?
3. Do you provide any instruction in calculus or analytic geometry?

SEQUENCE PATTERNS

From a tabulation of returns received from the thirty-five school systems, with Houston added as the thirty-sixth, it appears that most large city school systems follow a relatively uniform sequence for the mathematics courses offered during the first two years of high school. However, a marked variation in sequence appears for third- and fourth-year courses. As to range, offerings of calculus or analytic geometry were reported in ten cities, largely on an experimental basis, although some systems have been providing these courses in their regular programs for students of exceptional ability.

[1] The systems are those of Atlanta, Baltimore, Birmingham, Chicago, Cincinnati, Cleveland, Dallas, Denver, Detroit, El Paso, Fort Worth, Indianapolis, Kansas City, Long Beach, Los Angeles, Memphis, Milwaukee, Minneapolis, Newark, New Orleans, New York, Oakland, Oklahoma City, Omaha, Philadelphia, Pittsburgh, Richmond, St. Louis, St. Paul, San Antonio, San Diego, San Francisco, Seattle, Toledo, and Tulsa.

The college preparatory program in most of these big city schools begins with one year of elementary algebra (ordinarily offered in the ninth grade) and continues with a year of plane geometry, a sequence which must be completed before the student may proceed to more advanced mathematics courses. Of the thirty-six school systems covered in the survey, twenty-nine follow the algebra-plane geometry sequence in the first two years of high school: Baltimore, Birmingham, Chicago, Cincinnati, Cleveland, Dallas, Detroit, Indianapolis, Kansas City, Long Beach, Los Angeles, Milwaukee, Minneapolis, Newark, New Orleans, New York, Oakland, Oklahoma City, Omaha, Philadelphia, Pittsburgh, Richmond, St. Louis, St. Paul, San Diego, San Francisco, Seattle, Toledo, and Tulsa.

Nearly all of these large city schools offer a second year of algebra plus a single-semester course in solid geometry and another in trigonometry. Some of them offer in addition to these subjects a special course for seniors called "College Algebra," or an advanced concept course called "Mathematical Analysis."

Variations from uniform sequence

Varying from the usual pattern, four of these big city systems schedule two full years of algebra in the ninth and tenth grades, then one year of plane geometry in the eleventh and solid geometry and trigonometry in the twelfth. Cities reporting this schedule are Atlanta, El Paso, Memphis, and San Antonio. In San Antonio geometry may be taken after only 1½ years of algebra if this is advisable because of mid-term promotion.

In three other systems—Denver, Fort Worth, and Houston—1½ years of algebra are required before students may start plane geometry. In Fort Worth, it is reported that students are allowed to take two straight years of algebra in grades nine and ten if they want only the minimum two years of mathematics required for graduation, since a combination of algebra and geometry would require 2½ years. The normal sequence quoted in Denver is as follows: Algebra I, II, and III; Plane Geometry I and II; Algebra IV; then solid geometry and/or trigonometry. In Houston, solid geometry normally follows Plane Geometry I and II and precedes Algebra IV.

All these school systems offer a full program of third- and fourth-year mathematics, but the sequence in which these courses are offered varies somewhat. Nine of the school systems did not mention any definite sequence for advanced courses, but the indications were that second-year algebra normally is taken in the eleventh grade, with solid geometry and trigonometry offered in the twelfth. The systems that did not definitely state a sequence for advanced courses were Fort Worth, Long Beach, Los Angeles, New Orleans, Oakland, Oklahoma City, Pittsburgh, San Francisco, and Toledo. The response from Omaha definitely stated that there was no set pattern for courses that follow Algebra III in the eleventh grade. Los Angeles reported that a great deal of flexibility is allowed as to sequence of advanced courses.

Experimental programs

Several school systems have developed experimental programs. San Diego is reported to be conducting an experiment in one high school by giving algebra in the ninth and tenth grades and geometry in the eleventh. The response from Indianapolis indicates that pupils there are permitted to take second-year algebra along with geometry in the tenth grade if they desire—in fact, this is advised for those who are planning to take chemistry or physics in grade eleven, or who want to take the full five years of mathematics offered. Seattle schools are reported to have a course labeled "Laboratory Geometry" for their non-college students, along with an experiment being conducted among superior students in teaching solid geometry together with plane geometry in the tenth grade.

The Tulsa system reports that its students who are weak in mathematics usually take general mathematics in the ninth grade, algebra in the tenth, and geometry in the eleventh. On the other hand, it is reported two classes of exceptional students in Tulsa are taking algebra in the eighth grade and will continue with second-year algebra in the ninth and plane geometry in the tenth. The report states that tentative plans are for these groups to continue with solid geometry and trigonometry in the eleventh grade and to take calculus and/or analytic geometry in the twelfth. Seattle schools are also reported to have some students taking an accelerated program begun on an experimental basis in the eighth grade. As seniors, they are now doing special college-level work. Other specially qualified students are taking a course that will cover Geometry I, II, and III in one year.

For many years, the Seattle system has offered a course called "Mathematical Analysis" at the twelfth-grade level in lieu of the usual advanced algebra and solid geometry combination. This course includes work in differential and integral calculus, analytic geometry, college algebra, and trigonometry. The regular course in solid geometry is offered once a year for those who desire it, or twelfth-graders may choose instead a course in Senior Arithmetic for improvement in computational skill.

Placement of solid geometry and trigonometry

In nine cities, second-year algebra (intermediate and advanced) is scheduled for the eleventh grade, with solid geometry and trigonometry offered in the twelfth. The systems reporting this sequence were Birmingham, Indianapolis, Kansas City, Minneapolis, Newark, Richmond, St. Paul, San Diego, and Tulsa. It is reported that those students in Indianapolis who take second-year algebra along with plane geometry in the tenth grade may take solid geometry and trigonometry in the eleventh grade. Cincinnati's mathematics courses are organized so that the trigonometry course is begun in the latter part of the eleventh grade and completed in the first part of the twelfth grade. This acceleration permits the inclusion of college algebra in the latter part of the twelfth grade. The New York system is said to offer in some of its high schools a choice of second-year algebra in the eleventh grade or a combination of intermediate algebra and trigonometry in those two semesters.

In ten other systems, the normal sequence assigns intermediate algebra to the eleventh grade and advanced algebra to the twelfth. The school systems reporting such a division of third- and fourth-semester algebra are Baltimore, Chicago, Cleveland, Dallas, Detroit, Milwaukee, New York, Philadelphia, St. Louis, and Seattle. Seven of these systems report that they combine solid geometry with third-semester algebra in grade eleven and teach trigonometry in the twelfth grade along with fourth-semester algebra. The other three systems—Baltimore, New York, and Seattle—report a reversal of this procedure. They schedule trigonometry and intermediate algebra in the eleventh grade and offer solid geometry along with advanced algebra in grade twelve.

Special advanced courses

In addition to these universal offerings of second-year algebra, solid geometry, and trigonometry in grades eleven and twelve, it is reported that an additional advanced course called "College Algebra" is offered to seniors at Atlanta, Indianapolis, and San Diego, and in certain New York City schools. According to the response from Chicago, its third-semester algebra is called "Advanced Algebra" and its fourth semester "College Algebra." A course called "Mathematical Analysis" is said to be offered at Denver and Los Angeles as well as at Seattle. A new "honors" course offered for the first time at San Diego is reported to combine

trigonometry, solid geometry, and college mathematics in the twelfth grade.

Courses in calculus or analytic geometry are offered to exceptional students in ten of the cities reported. In New York, where special high schools accommodate rapid learners, returns indicate that courses in mathematical analysis, calculus, college mathematics, surveying, and navigation are provided. Offerings of calculus and analytic geometry are reported in three Baltimore high schools, enabling students there to complete five years of mathematics in four.

A five-year mathematics program in Indianapolis is reported to include both calculus and analytics. The same subjects are said to be offered to advanced-standing students in Cincinnati. The regular four-year mathematics program at Cincinnati also includes college algebra. One high school in Philadelphia is reported to include both of these advanced-concept courses in its accelerated programs, with four other Philadelphia schools also preparing similar schedules. The Oakland system reports that some instruction in calculus and analytics is included in its advanced-algebra course. San Francisco schools on the other hand offer calculus and mathematical analysis as senior electives.

Conclusions and apparent trends

Underlying the numerous local variations in mathematics placement and sequence among large city school systems, there appears to be a foundation of fundamental agreement from which several basic trends or practices may be noted.

A definite placement pattern has evolved for first-year algebra and plane geometry. A well-established pattern of placement and sequence seems to have been generally accepted for first-year algebra and plane geometry in the larger school systems. Even though this pattern is not universally followed, there appears to be a clear requirement that at least one year of algebra is basic to achievement in plane geometry and that plane geometry is prerequisite to a full understanding of most advanced-mathematics courses. Although some systems provide more than one year of algebra before offering geometry, the general consensus seems to be that this exceeds the minimum requirements. Undoubtedly other factors have influenced these variations, but these were not touched upon in the survey.

There is no uniform pattern of sequence for advanced-mathematics courses. The positions of intermediate- and advanced-algebra courses in relation to solid geometry and trigonometry have not been as clearly established. It is apparent that intermediate (third-semester) algebra is considered prerequisite to both solid geometry and trigonometry, but advanced (fourth-semester) algebra is not generally seen as essential to a mastery of these subjects. Many schools offer either or both courses as independent choices in semesters following Algebra III. There appears to be no consistent placement for these subjects other than a tendency to assign both solid geometry and trigonometry to the twelfth grade. Where an eleventh-grade placement is assigned, it is more often solid geometry that gets the earlier offering. There is no indication, however, that one is considered prerequisite to the other, since these subjects often are proffered on an "and/or" basis.

Mathematics programs are being enriched and extended to provide for all students. There appears to be a strong movement under way to enrich and extend the high school mathematics program by the inclusion of such advanced standing courses as college algebra, mathematical analysis, calculus, and analytics for the benefit of exceptional students. Largely experimental at present and sometimes limited to special schools for rapid learners, it seems to be gaining favor in large cities where industry is making a strong plea for mathematicians and where local colleges and universities offer exemption from freshman mathematics courses to

high school graduates with this training. Of similar significance is the trend in some areas to offer variations of regular courses designed especially to fit the needs of non-college students.

A flexible placement pattern is provided to facilitate subject coordination. In some cities, the placement of courses is made elastic enough to fit the requirements of science majors who plan to take physics or chemistry, even if this means "doubling up" on mathematics courses at one stage. This is indicative of the high degree of coordination that is being developed in many high school curricula and the recognition of guidance factors that go into their making. Elasticity is evident also in the report of the Tulsa experiment in which first-year algebra is being offered to eighth-graders of exceptional ability. This type of acceleration facilitates the enrichment of the curriculum by making possible a five-year mathematics program and an advanced-science program as well.

There is uniformity in basic requirements but wide freedom of choice in advanced work. The over-all pattern of practice as drawn from this survey seems to reveal an accepted definition of certain basic understandings as vital to all higher mathematics and related subjects, but the uniformity that prevails in early mathematics requirements gives way after the first two years of high school to a type of flexible offering that makes it possible for a school to fit the regular mathematics program to each student's particular needs and to the specific needs of the community. This area of flexibility is being increased by recent movements to augment and accelerate the basic program for those with exceptional ability.

Letters to the editor

Gentlemen:

Thanks for some good ideas on circle area formulas (J. M. Kingston, "Formulas Can Be Fun," January 1957). I tried these on a Solid Geometry class as a Plane Geometry review and they ate it up! Keep it up.

Sincerely yours,
Brother F. Spoeth
Riordan High School
175 Phelan Avenue
San Francisco 12, California

Gentlemen:

I want to tell you how interesting and instructive your recent five articles on the "Cutting of Squares" have been to me.

I got interested in them while reviewing geometry for summer school, and ended up by making cardboard models of most of them, and transferring these into plywood models. The work involved was relatively easy, and I found that extreme accuracy was necessary in only a few cases.

As I was not teaching geometry, I was able to adapt the simpler ones to the 7th and 8th grade in several ways:

(1) By writing the words "square," "rectangle," "parallelogram," etc., across *all* the different parts so that they could be read *only* when the appropriate figure had been assembled, the slower 7th graders gained an insight into the geometric forms.

(2) By drawing an actual square in colored crayon inside the assembled square, the pupil was guided in assembling and recognizing it. The other shapes were drawn on the same pieces in different colors.

(3) After doing (1) or (2), the pupils usually wanted to turn the pieces over and see if they could make the different figures with nothing to guide them.

I hope there will be more of this type article in the future, and putting them into book form seems an excellent idea.

Very truly yours,
Chester W. Hawley
Southwest Miami High School
South Miami, Florida

Cross-number puzzles: A teaching aid for classes of general mathematics[1]

LOUIS GRANT BRANDES, *Encinal High School, Alameda, California.*

A good idea for enrichment and motivation.

THE CROSS-NUMBER PUZZLE

THE COMMON CROSS-NUMBER puzzle, or cross-figure puzzle, as it is sometimes called, is very similar to the crossword puzzle. It differs in that digits are used to form numbers instead of using letters to form words. It is similar in that it includes a puzzle block, along with horizontal and vertical columns of items that are used to provide definitions or answers for filling in the block.

There is nothing new about cross-number puzzles. They were probably produced about the same time as crossword puzzles. However, cross-number puzzles, though easier to construct, have had none of the popularity of the daily newspaper crossword puzzles. Though a number of the puzzles have been published in educational periodicals and a few writers have included some of them in their books, use of the puzzles has been pretty much limited to recreational mathematics. The possibility of using these puzzles as a teaching aid was seemingly overlooked until very recently.

EXPERIMENTAL-CONTROL STUDIES CONDUCTED WITH CROSS-NUMBER PUZZLES

During the spring semester of 1956, twenty selected teachers from secondary schools in various sections of the United States participated in experimental-control studies with the cross-number puzzles. Fifteen of the teacher participants, from schools in eight different states, completed the studies and reported their results. One of the studies was conducted with seventh-grade sections, two with eighth-grade sections, and twelve with ninth-grade sections. The studies were conducted with the expressed purpose of determining if a difference of achievement in arithmetic could be associated with the use of cross-number puzzles.

The puzzle materials used in the studies included thirty selected cross-number puzzles on the fundamentals of arithmetic as usually covered in courses of general mathematics. These materials were offered to an experimental section, while a control section of similar pupils, taught by the same teacher, was continued on "the regular bill of fare." The initial and final achievements of the two sections were measured with standardized tests, and the differences of achievement noted.[2]

RESULTS

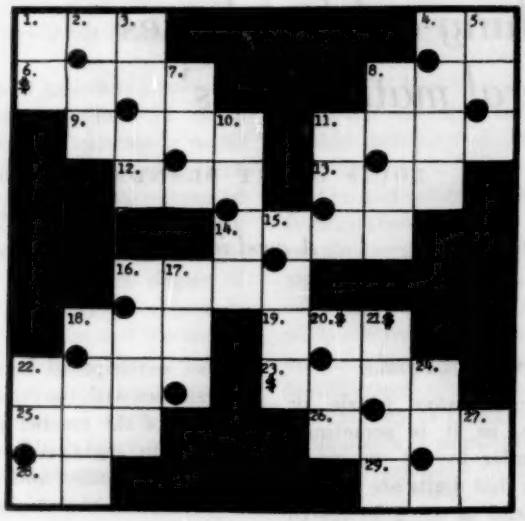
Here are the results of the study.

1. A comparison of indicated mean I.Q.'s, reading test results, size of sections,

[1] This article includes excerpts from *A Collection of Cross-Number Puzzles* by Louis Grant Brandes (J. Weston Walch, Publisher, P. O. Box 1075, Portland, Maine, 1957, 226 p., price: $2.50).

[2] Three of the studies made use of the Stanford Arithmetic Tests, forms J and K; twelve of the studies made use of the Arithmetic Tests, forms B and C. The latter tests have been standardized for use in the secondary schools of the Alameda Unified School District, Alameda, California.

Division of decimals



Horizontal

1. Divide: $7/\overline{8.89}$
4. Divide 9.48 by 12
6. Find correct to the nearest cent: $250.19 \div 8$
8. Divide: $2.4/\overline{173.52}$
9. Divide 16.416 by .6
11. Divide 5 by 1.6
12. Find the quotient correct to the nearest tenth: $2.7/\overline{36}$
13. Divide: $.3/\overline{.0012}$
14. Find the quotient correct to the nearest thousandth: $.96/\overline{1.76}$
16. Divide: $16/\overline{3.0000}$
18. Find the quotient to the nearest thousandth: $2 \div 3$
19. .25 of what number is 2.06?
22. 70.5 is .625 of what number?
23. If 5 tons of coal cost $62.15, how much will 8 tons of the coal cost?
25. In measuring the thickness of a steel plate, Jim took five readings from a micrometer. The readings were: .125; .127; .124; .128; and .126. Find the average of his readings in number of inches.
26. On a motor trip, a car travels a distance of 267.8 miles on 16.25 gallons of gasoline. How many miles per gallon did the car average on the trip?
28. At 16¢ a pound, how many pounds of apples can I buy for $4.48?
29. An airplane flew a distance of 508.5 miles in 2.25 hours. What was the average speed of the plane in number of miles per hour?

Vertical

1. At 57¢ per pound, what would be the weight of a piece of beef costing $7.41?
2. A scale on a chart is 1 inch equals five miles. How many inches are necessary to represent 10.6 miles on the chart?
3. Number of times 14 is contained in 1,017.94
4. Find the quotient correct to three decimal places: $2.5/\overline{18.061}$
5. $215.05 \div 2.3$
7. The result obtained by dividing 274.875 by 3.75
8. Divide: $48/\overline{3,409.44}$
10. Find the quotient correct to two decimal places: $9.3/\overline{587.49}$
11. $100.899 \div 3.33$
15. Divide 51.534 by 6
16. The result obtained by dividing 39.024 by 24
17. Number of times that .45 is contained in 395.1
18. Find the quotient correct to three decimal places: $45.961 \div 7.5$
20. A hot-rod club assesses each of fourteen members equally each month for anticipated club expenditures. If planned expenditures for December are $40.74, how much is each member's share?
21. A bill for 12 electric motors came to $535.44. Compute the cost of each motor.
22. How many hours will it take to make a trip of 520.8 miles in a car that will average 46.5 miles per hour?
24. Find the quotient correct to the nearest tenth: $1.9/\overline{83.89}$
27. Divide $325.08 by $3.78

and initial arithmetic-test results as reported by teacher participants indicated that the control sections of the studies "had much the better of it."

2. Nine teacher studies had experimental sections with a greater per cent of pupils improving in achievement than did the control sections; two studies had the same per cent improved for both experimental and control sections; four studies had a greater per cent of pupils in the control sections improving.

3. Ten teacher studies had experimental sections with a greater per cent of pupils improving their semester grades than did the control sections; five studies had control sections with the greater per cent of pupils improving their grades.

4. The differences in mean acheivements of eight studies were in favor of the experimental sections; seven were in favor of the control sections.

5. Comparisons made between pupils in each of the sections from four selected studies who had indicated I.Q.'s of 100 and above and those with indicated I.Q.'s below 100 showed small differences in achievement in favor of the low I.Q.'s from the experimental sections.

How to use puzzle materials

Ways to present cross-number puzzle materials to a class were suggested by the participating teacher. Their suggestions are summarized as follows:

1. As a class assignment for all pupils; the material for the course covered in regular classwork via the puzzles.
2. As a means of reviewing a short unit of work after it has been covered by a class.
3. As a reward for pupils who have completed the regular class assignment.
4. As supplementary material for individual pupils to work on at their own speed during class time and outside of class.
5. As a test after a similar puzzle has been used for review purposes.
6. As a general review before the end of a report card period and prior to an examination.
7. As a "change of pace" from regular classroom routine at regular or irregular intervals.
8. As a regular Friday review of work covered during the week.
9. As enrichment material for selected pupils to work on outside of class time.
10. As motivating materials for special days, such as days before school holidays, visiting days, etc., and at other times when the teacher must direct her interest to desk work or conferences.

Conclusions

In the opinion of the writer the following conclusions can be drawn from the results of the study:

1. Even though the ability levels of the experimental and control sections of the studies were in favor of the control sections, the experimental sections "more than held their own" and provided results in favor of using the puzzles regularly with classroom work.

2. The results indicate that the puzzles enabled many pupils to make greater improvement in achievement in arithmetical computation, and in their arithmetic marks, than those pupils who did not have an opportunity to use the puzzles.

If you have not used cross-number puzzles as a teaching aid for your classes, by all means plan to do so. The puzzles will delight the children, and you will be pleased with the results.

It is twenty-five years since Albert Einstein called attention to the fact that science could not produce happiness until man produced sense. That is why we had better be careful about what it is we ask of science. We are likely to get it.—*Taken from an editorial in* The Saturday Review, *October* 27, 1956.

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Recent discoveries in Babylonian mathematics, III: Trapezoids and quadratics

In the first two notes in this series[1] we discussed several really new extensions of our knowledge of Babylonian mathematics. In this note we will discuss a newly discovered tablet which gives another example of mathematical skills which, although previously known to have been a part of the Babylonian culture, may not be familiar to all of our readers—and which, in any case, would provide an interesting enrichment problem for our students.[2]

This problem associates the area of a trapezoid with simultaneous equations and quadratic equations. It occurs on a clay tablet 15.9×9.7×3.4 cms. along with two other problems and a list of coefficients which seem to relate to engineering problems associated with the use of mortar, bricks, and rammed earth in construction

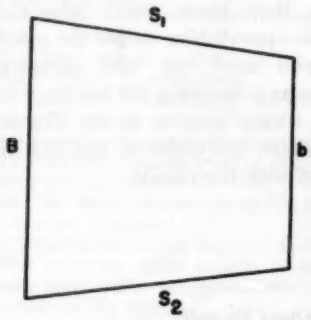


Figure 1

The problem is to find the length of the sides of an isosceles trapezoid (see Figure 1) given that the area is 150, that the difference of its bases is 5, i.e., $B-b=5$, and that its equal sides are 10 greater than two-thirds the sum of its bases, i.e., $S_1=S_2=\frac{2}{3}(B+b)+10$.

We might solve this problem by using the Pythagorean theorem to find the height of the trapezoid, and then using the area formula to get a quadratic equation in B and b from which we could eliminate B by letting $B=b+5$.

The Babylonians followed essentially this same procedure, *except* that they used an incorrect formula for the area of a trapezoid. They found the area as the product of the averages of the two pairs of opposite sides. (This formula is correct for some quadrilaterals. Can you find for which ones?) Thus they set the problem up as follows. (Of course, they used sexagesimal cuneiform numerals instead of Hindu-Arabic decimal-based numerals, and they used words instead of symbols or letters, and they actually wrote no formula but merely said "do this—do that" in such a way as to be equivalent to what we are writing.)

$$A=\frac{B+b}{2}\cdot\frac{S_1+S_2}{2}$$

(This formula is false, but is a fair approximation.)

[1] The Mathematics Teacher, vol. L (Feb., Oct., 1957), p. 162 ff., p. 442 ff.

[2] This is based on Taha Baqir, "Another Important Mathematical Text from Tell Harmal," *Sumer*, vol. VI (1950), pp. 130–148.

$$(1)\quad 150=\frac{2b+5}{2}\cdot\frac{2}{2}\left[\frac{2}{3}(2b+5)+10\right]$$

or, if we let $x=B+b=2b+5$

$$(2)\qquad 150=\frac{x}{2}\cdot\left[\frac{2}{3}x+10\right].$$

We would now clear of fractions and expand to get

$$(3)\qquad 150\cdot\frac{3}{2}\cdot 2=x^2+\frac{3}{2}\cdot 10\cdot x.$$

Completing the square we would have

$$(4)\quad 450+\left[\frac{1}{2}\cdot\frac{3}{2}\cdot 10\right]^2=x^2+\frac{3}{2}\cdot 10x+\left[\frac{1}{2}\cdot\frac{3}{2}\cdot 10\right]^2$$

$$(5)\qquad \pm\sqrt{506\tfrac{1}{4}}=\pm 22\tfrac{1}{2}=x+7\tfrac{1}{2}$$

$$(6)\qquad x=-7\tfrac{1}{2}\pm 22\tfrac{1}{2}.$$

The Babylonian followed identically these same steps except that he had no idea of negative numbers or two square roots. Hence, he found

$$(7)\qquad x=15\quad\text{or}\quad B+b=15.$$

From this he said

$$(8)\qquad \frac{B+b}{2}=\frac{15}{2}\quad\text{and}\quad\frac{B-b}{2}=\frac{5}{2}$$

and hence, by adding, $B=10$, and by subtracting, $b=5$.

Substituting in the given formula he completed the solution by finding

$$S_1=S_2=\frac{2}{3}\cdot 15+10=20.$$

It is interesting to note that he checked his work by verifying that these values of B, b, S_1, S_2 did give, by his formula, an area of 150.

In order that you may see a little more clearly exactly how the Babylonian proceeded without symbols or formulas, we'll quote what he wrote. After stating the problem he gave the following instructions:

(a) "Take the reciprocal of $\frac{2}{3}$; you get $\frac{3}{2}$."

(b) "Halve $\frac{3}{2}$; you get $\frac{3}{4}$."

(c) "Multiply $\frac{3}{4}$ by 150, the area, you get 225."

[Step (b) was a mistake. He did not need to halve $\frac{3}{4}$. The result of step (c), 225, is correct but would be obtained as $150\cdot\frac{3}{2}$ as in our step (3).]

(d) "Double 225; get 450. Keep it in your head."

[This completes the calculation of the left member of our equation (3).]

(e) "Take the reciprocal of $\frac{2}{3}$; it is $\frac{3}{2}$."

(f) "Halve $\frac{3}{2}$, get $\frac{3}{4}$. Multiply it by the 10 which you added, and get $7\frac{1}{2}$. Keep in your head."

(g) "Lay down (another) $7\frac{1}{2}$ and square it. You get $56\frac{1}{4}$."

(h) "Add $56\frac{1}{4}$ to 450 which your head has kept, and you get $506\frac{1}{4}$."

(i) "Extract the square root of $506\frac{1}{4}$. Its square root is $22\frac{1}{2}$."

(j) "From the square root, subtract $7\frac{1}{2}$ and 15 is the value of the unknown."

(k) "Halve 15, get $7\frac{1}{2}$."

(l) "Halve 5 by which one width exceeds the other, and you get $2\frac{1}{2}$."

(m) "Add $2\frac{1}{2}$ and $7\frac{1}{2}$; 10 you get. Subtract it from the second, 10 is the upper width and 5 is the lower width."

(n) [Check]

(o) "Such is the procedure."

It is on the basis of a number of such problems that we conclude that the Babylonians could solve quadratic equations with positive rational roots by processes which could have been extended to all quadratics even though they never did the extension nor used symbols nor stated general theorems.

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Emotional blocks in mathematics

by Mary K. Tulock, George Peabody College for Teachers, Nashville, Tennessee

Let us take an objective look at the attitudes of today's students toward mathematics. There is convincing evidence of the shrinking per cent of students studying mathematics. At the International Convention of Mathematicians, September 2–9, 1954, in Amsterdam, The Netherlands, Saunders MacLane, the former President of the Mathematics Association of America, reported on the status of mathematics in the United States. He reported on the de-emphasis of mathematics and the resulting impact on the greatly needed manpower supply. He contrasted this with the concerted effort for full development of mathematicians and other scientists within the countries behind the Iron Curtain. These countries, at the present rate, threaten to outstrip the United States by a wide margin in the future supply of mathematicians and other scientists essential today for physical survival. This awareness intensifies our concern but offers no solution to the complex problem. What *can* we do to improve the situation? Remove the threat of mathematics! Replace fear and trembling with appreciation. How? Why?

Let us look first at why. We, as people in a democratic society, are more interdependent than ever before in the history of man. We need to know more facts in order to survive. Each man holds the fate of others in his very hands. If you doubt this, pause to think a moment of the hazard Rip van Winkle would be today driving a car down Bay Shore Highway ignorant of our highly cooperative traffic regulations. This is but a small example of the need for dependable, intelligent citizens with self-assurance and self-direction.

What has this to do with mathematics? Much! We need men of confidence, courage, and skill. We do not increase man's confidence by robbing him of his sense of accomplishment. No man has succeeded through constant failure. Pupils who constantly fail mathematics have deflated egos and tend to develop attitudes of dislike and hostility toward mathematics. They do not willingly enroll in algebra classes because they are afraid—it represents a threat. People prefer success to failure. Why should they elect algebra? People tend to compete in a race only where there is a plausible chance of winning. They cannot fail—if they choose *not* to run. Many sacrifice a doubtful victory to prevent deflating defeat. Do percentage decreases in mathematics enrollments reflect this? What can we do about it? Or better, what should we do about it?

No real sense of accomplishment results from effortless success. Obviously the answer is not diluted mathematics. We will not solve the problem by de-emphasizing

mathematics, by lowering high school graduation requirements to one year in mathematics—or none. Then what?

Careful curriculum research such as is now being conducted at the University of Illinois under the direction of Max Beberman will reveal some specific answers. Educational theories are tested only in actual practice. Every alert and competent teacher constantly tests hypotheses—his own and others—through teaching, contemplating the consequences, analyzing the results, and speculating on possible improvements. Then what can we specifically do as teachers to provide mathematical competence for our students?

Among the numerous possibilities one in particular will catch the full focus of this paper. We can remedy existing emotional fear of mathematics and prevent future fear in so far as is humanly possible.

Fifteen years of dealing with emotional blocks in mathematics leave the author wondering about the classic remark "there is nothing to fear save fear itself." Some pupils with whom the writer has worked were so tense after traumatic experiences they seemed simply paralyzed. So rigid was the set against mathematics that some were convinced they just could not learn mathematics. Among the many who subsequently discovered that in actuality they *could* learn mathematics, at least six became so interested and intrigued with this power of mathematics that they themselves are now dedicated, successful mathematics teachers. So heady was the thrill of success in a previously hated and dreaded course that continued success awakened the desire to teach others that mathematics *can* be learned. How did these people recover?

One "patient" was a high school junior whose mother requested help from the writer, who was at the time a college professor of engineering mathematics. The mother's plea was so pathetic that an exploratory appointment resulted. This charming, shy girl simply needed her self-confidence built up, for she is actually brilliant. Painfully obvious at the beginning, her mathematical inferiority complex began to wane when she caught up with her class in algebra—after three weeks. Gratifying was the first glimmer of self-confidence when she solved her first word problem—without help! Thrilling beyond description was her assurance when she advanced *beyond* the class—and stayed out in front. Her mother reported a reflection of this assurance, self-direction, and poise in her daughter's entire attitude.

This girl—who, as a high school junior, in tears begged her mother (who is a highly educated lady) to let her stop school because she hated it and could not pass—is today a highly successful mathematics teacher.

How was this girl cured? (Let's call her "Bee.") First, the writer felt sure that Bee *could* succeed, and never once wavered in this belief. This assurance was carried to Bee in ways not entirely explainable—yet *sensed* by her and *known* for a certainty. Consistently failing in the past, Bee had resigned herself to expected failure and actually seemed unable to clear her mind of the poisons of fear and dread of failure. She "froze" and stuttered pathetically at first. The author felt deep empathy, but gave not the slightest indication of having noticed the stuttering.

Realizing the necessity of immediate success so as to start the ego-building process, the writer chose from arithmetic, "Bee, what answer would you get if you multiplied these two numerals together: $(5-2)\times(5+2)$?" She wrote $3\times7=21$. "Good!" was the matter-of-fact reply—indicating assurance that the correct answer had been expected. "Now let us look at these numerals more closely. We know the correct answer is 21. I wonder if we could arrive at the answer any other way." Pause! Bee stared at the board—started to speak—hesitated—and then said "I don't know." She had just succeeded. She did not want to risk failure, so she would rather not speak what she was thinking.

Choosing not to hazard a misunder-

standing on her part of the *negative* two involved, the writer deemed it wise to tread safe ground in this exploration of readiness, so Bee was given this problem: $(2+1)\times(2+2)$ gives what number as an answer? Bee wrote $3\times4=12$. Then Bee said "I think it would be correct to write $4+4+2+2=12$." "Splendid!" (With genuine pleasure evidenced.) This was the most hopeful indication yet. This revealed some basic knowledge of the underlying principles of algebra. The author pointed these out and substituted "pronumerals" for the number 2, saying "Algebra is really an extension of arithmetic." This transition she grew to understand.

She made rapid and permanent progress and her brilliant mind was unlocked and set free of fear, to operate with confidence and power.

In a similar way the writer helped to build the self-confidence of a star athlete who had a calculus block. Another case was a boy from the farm and a small rural school, who suddenly found himself competing with city boys in an engineering college of five thousand students. He was ready to go back to the farm, but was advised to try getting a tutor first. He recovered and is now a distinguished medical doctor. This doctor was serving in the Medical Corps in Tokyo in 1954. A captain on R & R from Korea met this young doctor, and upon learning he hailed from Virginia asked him if he knew the writer, since that is her native state. The young doctor then told the captain how he would never cease to appreciate the fact that his career was actually saved when he recovered from his mathematical block—which recovery he said was due to the author's help.

Many students in the writer's calculus classes have gained self-confidence partly because of the confidence the writer had in their ability. By expected success and provided-for success, by careful teaching methods and a genuine interest in our students we can build confidence in them rather than feelings of defeat and frustration. Granted, there are some pupils who should not take calculus—and we will not be able to provide beyond the accomplishment level of a pupil—but who is sure what this level is?

In a state college one young man was strongly advised not to enroll in engineering. The guidance office all but shut him out of the field of engineering. All indications were that he would fail. He announced "I have a right to try. How else can you prove I can not pass? I insist on a chance." With this spirit and determination he struggled through his freshman year of engineering subjects. He did not make distinguished marks—but he passed! He was undaunted and determined to attain his goal. His sophomore year was still rough but he started to pull up. The last quarter in his senior year he made honors! There are things to be considered—things within the very soul of an individual that are powerful factors in success or failure. These forces may not show on an entrance examination; they can be observed in an interview—they do operate—whether measured or not.

We as teachers are overlooking many of our able students. We need more guidance personnel versed in the knowledge of mathematics and the curriculum in addition to their guidance techniques. Evidence indicates we have an insufficient number of such people.

We need a mathematical attitude scale to facilitate the work of the guidance personnel and the classroom teacher. Any insight into the emotional turmoil which surges through frightened students will help alleviate the trouble. The writer has experimented in developing such a scale.

How would such a scale be useful? At the seventh-grade level the teacher would profit from evidence of dislike for mathematics on the part of students. He can make a special effort to build confidence in such pupils, explore the troublesome areas, and, through individual attention, maybe only a few minutes a day, help a

child grasp the understanding necessary for success.

Because grade seven is a critical and highly important level in the mathematical curriculum (as is also grade eight), a scale is being constructed to measure attitudes of these pupils. The author's opinion is that many of our potential successes in mathematics are lost at this crucial time. They dislike mathematics and they will register in grade eight for the simplest course in mathematics available. Today as never before we need these people in the field of mathematics. Many will be blocked from pursuing a science or engineering career because of a poor attitude or an emotional mathematical block. They can enter an engineering college without science and be much less handicapped than if they have only two years of mathematics, or less. The loss usually amounts to one or more years' time—expensive college time! We educators must rally to the cause and act. Recognizing the need is not sufficient, although it is necessary. The need is admitted. Now we must *do* something about it rather than lament in paralysis.

The author believes emotional blocks which should have been prevented—but, alas, were not—can and should be decreased or removed. After grade seven, or eight, it is most frequently too late.

The following, designated by the author as "Operation Fifteen," have proved successful many times.

It is possible to lessen or remove unfavorable attitudes toward mathematics by teaching methods that give individual attention to the child's feelings:

1. Set individual goals so that it is reasonable to expect that more mathematical experiences will culminate in feelings of success than culminate in feelings of failure.
2. Frequently judge the performance ability of each pupil and gear individual expectancy so as to make possible numerous initial rewards for successful accomplishment in the form of verbal confirmation of attained goals, good grades on exercises, successful class performance, etc.
3. Strive for interest by using puzzles, games, anecdotal materials, mathematical contests, audio-visual aids, resource materials from the environment, etc., that have a mathematical content or implication.
4. Praise, instantly and sincerely, accomplishments that are reasonably deserving of praise.
5. Discover as much as possible about *how* the child performs mathematical exercises.
   a. Such an analysis might reveal troublesome areas of misunderstanding.
6. Teach for real understanding of mathematical manipulations and evaluate understanding:
   a. By having student explain why and how he performed as he did in particular exercises.
   b. By challenging him with problem situations requiring him to analyze what the problem is, the probable solution, different methods that may possibly lead to the solution, the choice of a particular method that will likely lead to the solution, the use of the particular method, and the testing of the solution. At each stage the child should be able to give a lucid explanation of why, not just a rule of thumb.
7. Gradually draw the reticent child into the group activities by:
   a. Oral answers in class—being reasonably sure that he knows the answer before you ask him:
      1) If he raises his hand designating a willingness to answer.
      2) If he has previously displayed a knowledge of the fact in question.
   b. Volunteer board work.
   c. Material voluntarily brought in for class use.

d. Special report on a challenging question.

8. Convey to the child in the best possible way your interest in him and your concern for his welfare.
9. Show your genuine pleasure at outstanding success and improvement.
10. Register disappointment, but cautiously, in situations where you are reasonably sure that the child has not performed up to his ability.
11. Keep the work challenging so that the child can feel a sense of accomplishment and satisfaction rather than humiliation.
12. Avoid criticizing the child either in public or in private.
13. Avoid being sarcastic.
14. Avoid being critical of the child's mathematical efforts except in a constructive way.
15. Avoid being patronizing.

Have you read?

NEWMAN, LAWRENCE. "Meaningful Mathematics Problem Work for the Deaf," *American Annals of the Deaf*, May 1957, pp. 293–299.

Although most of us are not in positions of teaching mathematics to the deaf, this article will be of much interest. The problems encountered and the purposes of instruction are nearly the same for all groups. Much of the research on solving verbal problems is applicable to the non-deaf as well as the deaf. The use of concrete materials is important. The emphasis on meanings of words appearing in the mathematical senses, as well as the relationship between the abstract ideas of mathematics and the concrete situations of life, must be emphasized. A teacher's treatment of direct and vicarious experience is very important in making the mathematics problems functional. You will be interested in comparing the methods used with the deaf to those you use. This article adds some insight into the teaching of verbal problems.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

PIHL, JACK L. "Taking the Stock Market into the Classroom," *American Business Education*, March 1957, pp. 175–178.

Many of you may do some teaching about investments either in junior high school arithmetic or in general mathematics courses. If you do, this article is a must for you. It presents a method used in Indianapolis by a teacher and a broker. Here is an excellent example of pupil, teacher, and layman planning together to better prepare boys and girls for the future. Mr. Pihl of the brokerage firm gives a step-by-step outline of how the interest in the program was developed, how the specific plans were laid, how the teacher, pupils, and brokerage firm made their special contributions. It is interesting to note the enthusiasm of all concerned in the project. This method also has implications for other areas of study similar in nature.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SCHMIDT, PAUL F. "Models of Scientific Thought," *American Scientist*, March 1957, pp. 137–149.

Much lip service has been paid to methods of scientific thought. The author of this article points out the historical development and what brings about scientific thought.

The Greeks had two essential characteristics of scientific thought, namely: demonstration without empirical observations and applicability to nature. The first characteristic gave the impetus to Plato, Euclid, Pythagoras, and they added the essential characteristics of interconnected propositions, a deductive system, axiomatic, true of nature, systematic, and universal.

But Socrates saw the weakness in this system—namely, the basic assumptions where agreement might be hard to obtain. Therefore, he assumed but did not accept.

Aristotle contributed the idea that sense experience is a source of scientific knowledge.

Copernicus pointed out that often two different hypotheses could result in the same conclusion. Descartes of course followed Euclid but tried to find a better starting point.

The second great model after Euclid was Newton, who declared that justification of a proposition is deductive from consequences, and that it can be tested only by probability.

Another big step was that of Hilbert, around 1900, when physical and formal geometry seemed two different things, but soon Einstein propounded his theory of relativity and needed the extended geometry.

A review such as this hits only the high spots, but you and your students will find this interesting and profitable reading.—PHILIP PEAK *Indiana University, Bloomington, Indiana.*

A column of unofficial comment

The Jeffrey report on Euclidean geometry

by Howard F. Fehr, Teachers College, Columbia University, New York City

A full year of study of Euclidean synthetic plane geometry and another half year of solid geometry has been the established pattern in the schools of the United States for the past one hundred years. There have been changes in sequence, of theorems in the list of assumptions (postulates or axioms), and in the degree of rigor, but the pattern still exists today. At times, efforts have been made to change this pattern, to fuse the study of solid synthetic geometry with the plane geometry to stress logic in everyday life, to introduce algebra and some analytic geometry, but the efforts have had little effect on the established pattern.

The time has now come when we must seriously question the value of the one and one-half years of geometry as desirable in an up-to-date mathematical program. This is not to deny the value of the study of synthetic geometry, but it is to question the value of the length of time allotted to this study. Deduction, as a mathematical mode of proof, has equal value in arithmetic, algebra, and analytic geometry. Moreover, analytic geometry has far greater potential in subsequent study of mathematics and applied sciences than does synthetic geometry. Further, the deductive aspects of the usual high school solid geometry have little value as deduction, while the facts about solids and their mensurational properties have much value.

It is seriously proposed that the study of synthetic geometry be limited to the proofs of a very few but important theorems, that proofs of originals be based on these theorems and other theorems proved as originals, and that the study lead as quickly as possible—certainly within 4 months—to the proof of the Pythagorean Theorem. This theorem leads naturally to the introduction of distance and coordinate geometry; the study of coordinate (analytic) geometry can then be extended side by side with synthetic geometry in developing the properties of plane figures. Natural extensions to solid geometry (three-dimensional figures) can be made intuitively. This abridgement of the study of geometry permits the introduction of the study of trigonometry through coordinates both rectangular and polar and thus gives unity to mathematics—algebra, geometry (analytic and synthetic geometry), and trigonometry become one branch of study.

To support this proposal of an abridged Euclidean and more informative study of geometry there follows an excerpt from the Jeffery British Report which is now the syllabus controlling School Certificate Mathematics for English Grammar Schools (comparable to U.S.A. College Preparatory Programs). It should be read by all teachers as a trend and direction in the teaching of secondary school mathematics.

Excerpts from:

Report of a conference of representatives of examining bodies and teachers' associations with a suggested alternative syllabus and specimen papers

The Cambridge proposals, besides encouraging the fusion of geometry and trigonometry, remove much of the emphasis from formal work, but the list of theorems is still too long. Since the fundamental theorems about congruence, similarity, elementary trigonometry, and elementary solid geometry ought to be presented to pupils on an informal basis, no proofs of these results should be demanded in examinations. Outside this basic work, there are certain key-theorems, and a number of other theorems that can be derived from them. Proofs of the key-theorems, and deductions of the others, form an important part of the geometry teaching, but it is doubtful whether any proofs of theorems should be required in examinations. If all the key-theorems and derived theorems are liable to occur, too much teaching time will be sacrificed to their preparation. The theorem-work in examinations might well be limited to the key-theorems: or a few other theorems might be included to illustrate diverse methods of proof. The proposed list of "italicized" theorems is not necessarily the best possible selection, but it is of about the right length if there are to be theorems at all. There is room for discussion among teachers about the ideal list of "italicized" theorems, and it might be useful to compile a list of derived theorems (to be available for rider work, but of which proofs would not be required in the examination). It is unlikely that all the examining bodies will make precisely the same classification of theorems. There should not be more than one theorem in any paper, and it should come in the non-compulsory section. This restriction would prevent undue concentration on this part of the work.

Two dimensional figures

A sound appreciation of the properties set out below will be expected. Proofs of the items in *italics* may be required. Proofs of the other properties will not be required. Riders may be set which can be solved by the use of these properties, but they themselves will not be set as riders. In solving riders candidates may use any knowledge they possess. Candidates will be expected to understand the relation between a theorem and its converse.

Properties of angles at a point and angles made with parallel lines.

The exterior angle property and angle-sum of a triangle.

Angle-sum properties of polygons.

Congruency of triangles: similarity of triangles.

Symmetry about a point or line.

The isosceles triangle: the parallelogram, rectangle, square.

Parallelograms on the same base and between the same parallels are equal in area.

The *straight* line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side. (By congruence or areas or similarity.)

Other area properties of rectangles, parallelograms, triangles, and trapezia, including the formulae

$$\tfrac{1}{2}bc \sin A \text{ and } \sqrt{s(s-a)(s-b)(s-c)}.$$

Connections between algebra and geometry, e.g., $(a+b)^2$, use of coordinates, including negative coordinates.

The sine, cosine, and tangent of an angle, acute or obtuse.

Trigonometry of obtuse angles will not occur in the first section of the paper.

The sine-rule for a triangle.

Numerical applications involving the use of four-figure tables but not including the ambiguous case.

The theorem of Pythagoras. (By proving that the perpendicular drawn to the hypotenuse divides the triangle into similar triangles, or otherwise.)

$$\sin^2 A + \cos^2 = 1$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This replaces the "extensions of Pythagoras": easy numerical applications using four-figure tables will be set.

The symmetrical properties of chords of a circle.

An angle at the center of a circle is twice any angle at the circumference standing on the same arc.

The other "angle properties" of the circle.

The perpendicularity of tangent and radius; the distance between centers of circles in contact; the equality of tangents from an external point.

The "alternate segment" theorem.

The "intersecting chord" theorem for an external point ($OP \cdot OQ = OR \cdot OS = OT^2$).

The analogous property for an internal point.

The relationship between the areas of similar triangles.

Corresponding results for similar figures and extension to volumes of similar solids.

The bisector of any angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

The analogous property for an exterior angle.

Knowledge of simple loci, with easy extensions to three dimensions. The method of intersecting loci. Plotting of simple loci other than the circle, e.g., parabola or ellipse.

Knowledge of the following "ruler and compasses" constructions will be assumed: set of squares will be allowed for the construction of parallel lines.

Bisection of angles and straight lines.

Construction of perpendicular to a given line and of angles equal to a given angle.

Construction of angles of 30°, 45°, 60°.

Construction of triangles, quadrilaterals, and circles from simple data, including the inscribed and circumscribed circles of a triangle.

Division of a straight line into a given number of parts or in a given ratio.

Construction of a triangle equal in area to a quadrilateral or pentagon and a square equal in area to a given rectangle.

Construction of tangents from an external point.

Construction of a segment containing a given angle.

THREE DIMENSIONAL FIGURES

The properties of parallel planes; the normal to a plane, the angle between two planes, and the angle between a plane and a straight line.

The forms of the cube, rectangular block, pyramid, tetrahedron, prism, circular cylinder, circular cone, and sphere (including calculations of lengths and angles).

PRACTICAL APPLICATIONS

In addition to ordinary riders, questions involving drawing, trigonometry or geometrical reasoning may be set on the following topics:

(a) Simple map problems, scales, contour lines, slopes.

(b) Determination of positions by two bearings; the nautical mile and knot; the triangle of velocities.

(c) Heights and distances.

(d) Simple plan and elevation problems.

(e) Latitude and longitude: great and small circles on a sphere.

(f) The length of an arc in terms of the radius and the measure (in degrees) of the angle of the center.

(g) Three dimensional problems which can be solved by analysis into plane figures.

What's new?

BOOKS

MISCELLANEOUS

Niels Henrik Abel, Mathematician Extraordinary. Oystein Ore. Minneapolis: University of Minnesota Press, 1957, Cloth, 277 pp., $5.75.

A Survey of Research Potential and Training in the Mathematical Sciences. G. A. Hedlund, Department of Mathematics, Leet Oliver Memorial, Yale University, New Haven, Connecticut. A report of the Conference on Undergraduate Mathematics Curricula held at Hunter College in October, 1956. 71 pp., free.

INSTRUCTIONAL MATERIALS

Income Tax Teaching Kits. Internal Revenue Service, Washington 25, D. C. Kits available in both general and farm types, free.

Is "Math" in the Stars for You? (Catalog No. S 1.69:65), Superintendent of Documents, Government Printing Office, Washington 25, D. C. 6-page booklet dealing with professional opportunities for girls interested in mathematics, 5 cents.

Protractangle. Green-Way Novelty Company 1031 15th Street, Denver, Colorado. (Also sold through Safeway Stores and Woolworth's Stores.) Plastic drawing instrument, $1.25.

Publications Resulting from National Science Foundation Research Grants Through Fiscal Year Ending June 30, 1956 (Catalog No. NS 1.13: G 76). Superintendent of Documents, Government Printing Office, Washington 25, D. C. 38-page booklet giving more than a thousand references to publications stemming wholly or in part from research grants of the NSF, 30 cents.

A Study of the Science and Mathematics Courses elected by the 1956 Senior Class, and the Number of Seniors who planned to specialize in scientific fields in the Academic High Schools in New York City. Samuel Schenberg, Supervisor of Science, Board of Education, New York City, New York. 31-page booklet, free while supply lasts.

FILM

(A PLUS B) Squared (No. 1X178). International Film Bureau, Inc., 57 East Jackson Blvd., Chicago 4, Illinois. B. & w. with sound; 10 min. Rental, $2.50; sale, $50.00.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Algebra: Its Big Ideas and Basic Skills, Book I (second edition), Daymond J. Aiken, Kenneth B. Henderson, Robert E. Pingry. New York: McGraw-Hill Book Company, 1957. Cloth, xiv+434 pp., $3.28.

Algebra: Its Big Ideas and Basic Skills, Book II (second edition), Daymond J. Aiken, Kenneth B. Henderson, Robert E. Pingry. New York: McGraw-Hill Book Company, 1957. Cloth, xiv+434 pp., $3.28.

These two books, one for the first year and one for the second year, carry the same titles as the earlier editions by the same authors. The titles give a clue to the emphasis and the pattern of organization. The authors have deliberately identified the relatively few "big ideas" in the study of algebra. They constantly keep these before the student so that as he learns the "basic skills" he will not lose sight of the "big ideas." Six of these are listed in Book I: letters for numbers, exponent, equation, number, function, graph. In Book II, there are seven, which include those in the first book, though the idea in the second may be more comprehensive than that in the first (i.e., "variable" instead of "letters for numbers").

The authors have done what appears to be a good job of identifying for the student what he is supposed to be learning. They place in boldface type a statement of what is to be learned and label it "Your Aim." Then at the end of each chapter these aims are restated and review problems are given to test each.

New ideas are developed throughout each book by asking the students questions that lead him from the familiar progressively closer to the new idea. After the student has had a chance to generalize for himself the rule or principle is stated under the label "Do you see?"

There is adequate drill material to fix both the "ideas" and the "skills." There are plenty of easy exercises, and, in most problem sets, there are decidedly more difficult problems at the end which should challenge the more able students. There is a good selection of problems expressed in words throughout both books. Self-tests are found frequently in the books. At the end of each chapter there is a chapter test, a set of cumulative review problems, and practice problems in arithmetic skills. There is a real attempt to test understanding as well as skills throughout the books, and though this is difficult I think it is something we should try to do.

The first book starts almost immediately to use letters for numbers and to perform the fundamental operations with letters. It ties these operations up with arithmetic and uses them to strengthen both the skills and understandings in arithmetic. An example is the examination and statement of the commutative principle in multiplication found on page 17. The exercises have a few problems where the arithmetic is not trivial (especially in solving linear equations), so that the student has a chance to maintain his skill with fractions, decimals, per cents, etc. The word problems involving units and tens digits are illustrated with pictures of bundles of tens. There are several indications that the people writing this book know what is going on in the current teaching of arithmetic and are willing to build on it. I think this is good and feel they could do even more of it.

The second chapter deals with equations and the third with signed numbers. There are two good chapters (6 and 7) on functions and graphs and special products, and factoring is postponed to chapter nine. This is in line with the current trend to de-emphasize the purely manipulative aspects of algebra. The rather standard chapter (13) on quadratic equations is followed by one on indirect measurement which introduces three trigonometric ratios.

The review materials in the second book are more than a review. They seem to be what the authors call them: "A new look" at various topics. As in the first book, the contents of the second book are the traditional topics.

These two books do a good job of presenting the traditional algebra course. They follow the accepted recommendations as far as emphasis is concerned. They do a much better than average job of developing new material and emphasizing basic understandings. For those who know the first edition of these books, may I add that the second edition is much more than a minor revision. The development of new material and the explanations are much improved though the fundamental idea of stressing "Big Ideas" is retained.—*Isobel Blyth, Michigan State University, East Lansing, Michigan.*

Algebra, Second Course, John R. Mayor and Marie S. Wilcox. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957. Cloth, vi+458 pp., $3.92.

This text is a definite step in the direction of a presentation which emphasizes the structure of algebra rather than its manipulative aspects.

The language used is for the most part close to the language of modern mathematics. The conventional topics of second-year algebra are adequately presented. Throughout the text great emphasis is placed upon graphical methods.

The three last chapters deserve particular mention. Chapter 11 deals with Statistics and Probability. The treatment of statistics is standard. Students are introduced to such concepts as mean, median, mode, percentile, standard deviation, binomial distribution, and normal frequency curve. There is a good discussion of empirical probability. It is difficult to see how much more could be included in an introductory chapter on statistics for high school students.

Chapter 12 discusses the Nature of Proof in Algebra. Mathematical induction is presented here. Postulates are listed which describe some of the important properties of the algebras of the real numbers and the rational numbers. Several theorems are proved from these postulates. It is to be hoped that teachers will give careful attention to this section of the chapter. On pages 358 and 359 is found an acceptable set of postulates characterizing positive numbers. Proofs are given that if a and b are positive then $a(-b) = -(ab)$ and $(-a)(-b) = ab$. The last part of this chapter contains some material on the algebra of sets. This should not prove too difficult to teach.

The final chapter deals with curve sketching. Careful coverage of this chapter would lay a sound foundation for a course in analytic geometry.

Refreshingly few rules are given to hamstring student initiative and ingenuity. Chapter tests and cumulative reviews at the end of each chapter are helpful teaching aids. Many of the exercises review geometric concepts. In general, problem lists seem quite adequate.

Discussions of such key terms as constant, variable, function, independent variable, ratio, etc. are essentially correct.

It is obvious that the authors are familiar with modern interpretations of basic mathematical concepts. The phrase *literal number* is not to be found in the text. Variables and constants are defined satisfactorily. Most high school algebra texts contain meaningless discussions of concepts which cannot do other than confuse students and teachers. This text will give rise to relatively few misconceptions.

There are a few errors. There are some places where the authors use language which is vague and unsatisfactory. Attention will be called to some of these items below. Of course, this reviewer would much prefer that some of the postulates for the algebra of the complex-number system be listed early in the text and then used in the development of many topics. This might be a rather daring innovation to make at the present stage of the reformation of high school mathematics.

The distinction on page 5 between a conditional equation and an identity is not made as clearly as might be done. There is a misprint on page 56 where it is said, "The roots of the equation $x^2+9=0$ are written $x = \pm\sqrt{-3}$." The introduction of complex numbers at this point seems much too casual. It would be well to observe that the equation $x^2+3=0$ is not satisfied by any positive number, negative number, or zero. Hence, when we say that the symbols $\sqrt{-3}$ and $-\sqrt{-3}$ represent roots of this equation we are assuming that numbers can be invented which are neither positive nor negative. If we had available at this point the careful discussion of positive and negative numbers which occurs in Chapter 12 we could point out that the numbers $\sqrt{-3}$ and $-\sqrt{-3}$ satisfy the *relation of being negatives of each other* but neither number possesses the *property of being negative*. Of course the complex numbers must remain shrouded in mystery for students until they are privileged to see how these numbers may be *constructed* from the real numbers.

On page 57 we are asked to *give the equivalent of* $3\sqrt{-16a^2}$. The answer is given as $12ai$. It would be better to use our definitions to point out that for every positive number a we have $\sqrt{a^2} = a$ and for every negative number a we have $\sqrt{a^2} = -a$. In the same fashion $3\sqrt{-16a^2} = 12ai$ or $-12ai$ according as a represents a positive or negative number.

The chart on page 60 describing the structure of the complex-number system implies that integers and fractions are disjoint subsets of the set of rational numbers.

The chapters on exponents, radicals and logarithms are carefully written. The introduction of fractional and negative exponents is satisfactory.

The section on determinants in the chapter on Systems of Equations is good. There is no pretense of presenting general proofs of the difficult theorems of determinant theory. These theorems are simply illustrated and used. There are some logical weaknesses in this chapter. A determinant is called an *array of numbers which has a value*. Is a determinant a symbol or a number? In general this text does not distinguish carefully between the symbols of mathematics and the things that these symbols represent. There are essentially four concepts to be developed in an introductory discussion of determinants. On the one hand, we consider ordered sets of numbers that are called matrices. We introduce symbols to represent these matrices. With certain matrices we associate numbers which are called the determinants of these matrices. We must have symbols to represent these determinants.

The discussion of constants and variables on page 247 is excellent. Constants and variables are carefully described as symbols used to refer to sets of objects. A slight slip occurs here. It is stated that *the range of a variable is the collection of constants which may be substituted for it*. The authors do not really mean this as they show by their examples. In the same paragraph where the above definition occurs the variable "a student" is employed in an example, and it is pointed out correctly that the range of "a student" is the set of all students in a certain high

school. For example, we may say "if *a student* is tardy he must report to the office." The variable (symbol) *a student* may be replaced by certain constants (symbols) like *John Jones* and *Paul White*, names of individual students. We say that the *range* of the variable is the *set of students* rather than the *names* of the students in the set.

The discussion of dependent and independent variables, functional notation, direct and inverse variation is excellent. Many traditional errors are avoided in these pages.

In Chapter 12, The Nature of Proof in Algebra, there are some logical weaknesses. For example, the fifth multiplication postulate asserts that: *For every element $a \neq 0$ there exists a corresponding element $1/a$ such that $a \cdot 1/a = 1$.* It would seem well either to *prove* that for each element a the element $1/a$ is unique or else to *postulate* this uniqueness. Of course the proof is easy, for if $a \cdot x = 1$ then $a \cdot 1/a = a \cdot x$ and $1/a(a \cdot 1/a) = 1/a(a \cdot x)$ and, using previous postulates, $1/a = x$. Some remarks at the bottom of page 355 and the top of page 356 indicate that the authors look upon the assertion, if $a+b = a+c$ *then* $b = c$, as a more general assumption than the other postulates listed. Actually this is only a trivial theorem. For

if	$a+b=a+c$
then	$(-a)+(a+b)=(-a)+(a+c)$
and	$((-a)+a)+b=((-a)+a)+c$
and	$b=c.$

It must be that the authors have in mind certain postulates of mathematical logic which are sometimes used without comment. For example, if *equality* is taken as an *undefined relation* then we may postulate that it has certain properties, such as, for every a, b, c it is true that:

1) $a = a$
2) if $a = b$ then $b = a$
3) if $a = b$ and $b = c$ then $a = c$.

In mathematical logic these three assertions above may be derived as theorems from other assumptions. But in elementary algebra we might as well define the relation "$=$" precisely by observing that when we assert $a = b$ we mean simply that a and b are symbols for the same object. This simple definition obviates the necessity for such statements as "things equal to the same thing are equal to each other."

In summary, we wish to re-emphasize that this text is noteworthy in that the terminology employed is for the most part simple and correct. Teachers will find the problem lists satisfactory. There is no radical revision of content. Some new material on statistics, set theory, and the deductive nature of algebra is available. This is in line with recent recommendations that have been made by many mathematicians. However, it is not true that concepts of set theory and principles of deductive proof are used as unifying concepts. We are still waiting for such an algebra text.—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana.*

Calculus with Analytic Geometry, Richard E. Johnson and Fred L. Kiokemeister. Boston: Allyn and Bacon, Inc., 1957. Cloth, xi+650 pp., $7.95.

The material covered in this text is not unusual for current texts in analytic geometry and calculus, and the previous preparation assumed is also traditional. The outstanding feature of this text is the presentation of the calculus for functions of one variable in a complete and rigorous way. This is certainly a worthwhile contribution, and the authors are to be warmly congratulated for having produced it. The attitude of decent respect for mathematical truth which one encounters in this text is as refreshing as it is rare in elementary textbooks. Nonetheless, the beauty of the authors' exposition of the basic facts of calculus for functions of one variable is, in the opinion of this reviewer, somewhat obscured by the inclusion of a large number of topics whose presence seems to be best explained only by tradition. The usual prolixity of physical applications, the necessarily incomplete treatment of the calculus for functions of more than one variable, and the final "brief treatment" of differential equations are examples of the glaring pedagogic weaknesses of the traditional calculus course which are left uncorrected in this text.

The main interest for high school teachers of mathematics in a text such as this is its possible use as enrichment material for the gifted students. In spite of the traditional shortcomings we have noted, the mathematical content of this text as compared with that of most currently available texts justifies this reviewer in recommending the present text very highly for just such use.—*M. F. Smiley, State University of Iowa, Iowa City, Iowa*

Fundamental Mathematics, Leslie H. Miller. Henry Holt and Company, New York: 1957. Paper, vi+323 pages, $3.50.

This is a text and workbook. Space is provided for working and answering exercises within the body of the book, and 82 pages of tear-out exercises are placed in the back of the book. Those exercises cover the twelve chapters of the text, averaging three sets per chapter. The scope of the text is moderate, beginning with a review of arithmetic, extending as far in algebra as quadratic equations, a brief treatment of complex numbers and logarithms, including some work on progressions, and ending with a short preview of trigonometry.

The topics are treated in quite a traditional manner, yet they are treated well in the sense that teachability and learnability were uppermost in the author's mind. It was written for use in that no man's land of remedial mathematics for college students, usually taken at the freshman level. (In his preface, Miller states that his -

purpose is to develop knowledge and skill in handling quantitative problems that arise in the pursuit of a college education.) Given its conventionality, all in all, this is a good book. The arithmetic review in the first five chapters is handled in a mature and interesting way, with a nice balance between logical and intuitive presentation. Different number bases are used here to good advantage. Later, a chapter on variation, though brief, satisfies the need for the general college student to know some of the language of variation. The emphasis on graphs proceeds from general information graphs through those of simultaneous equations, both linear and quadratic. It is unfortunate that no use is made of the function concept in this text, especially here in graphs.

There are some places where errors or "unfortunate" ways of expression occur. For example, "larger" as used on p. 11 implies that -8 is larger than 3. On p. 130, the definition of principle nth root can lead an unlettered reader astray with the phrase "multiplied by itself n times." The rule on p. 63 for locating the decimal point in a product is wrong, for it refers to the "sum of the digits" rather than to the "sum of the number of digits" in the factors. In spite of an attempt to encourage careful grammar in the students, the author uses "data" throughout Chapter 9 as if it were a singular noun.

The errors just indicated do not seriously detract from the good qualities of this book, probably because they are not extremely numerous. In sum, this is a good, teachable, albeit conventional, approach to elementary mathematics for the general college student with a weak high school mathematics background.—*Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio.*

Linear Algebra for Undergraduates, D. C. Murdoch. New York: John Wiley and Sons, Inc., 1957. Cloth, xi+239 pp., $5.50.

Dr. Murdock has written a modern algebra text that satisfies a definite undergraduate need. Linear algebra, theory of matrices, and quadratic forms are treated. Although algebraic proof is emphasized, the reader is never allowed to wander too far from the concrete geometric interpretation of the algebraic statements. This should prove to be a big help for the beginning student who is not as yet sufficiently sophisticated or mathematically mature to absorb modern algebra at its most abstract.

This text is written in a good clear expository style. There are many illustrative examples to point up the definitions and the theorems as well as a good deal of interesting problem material. The transition from two- and three-dimensional to n-dimensional concepts is smooth. There is usually a preliminary discussion as to where a chapter or a section is going. At every stage there is motivation for the definitions that are made and as a result the reader obtains a feeling for the unity and reasonableness of mathematics as well as a good idea of how the mathematician operates in generalizing concepts.

This book is not too difficult for the serious undergraduate who has had college algebra, analytic geometry, and a little extra time for these courses to settle. There are some excellent tips to the student on how to read algebraic material which should enable him to ease himself into the reading of algebra as painlessly as possible.—*Augusta Schurrer, Iowa State Teachers College, Cedar Falls, Iowa.*

Understanding Arithmetic (revised edition), Robert L. Swain. New York: Rinehart and Company, Inc., 1957. Cloth, xxi+264 pp., $4.75.

This is a professional book dealing primarily with the content of arithmetic rather than with instructional method. It is intended for use chiefly in the pre-service preparation of elementary school and junior high school teachers but may be used to advantage with certain other student groups as well.

A cursory examination of Swain's *Understanding Arithmetic* reveals a similarity to books such as Buckingham's *Elementary Arithmetic, Its Meaning and Practice* and to Mueller's *Arithmetic, Its Structure and Concepts*. This certainly is true from the standpoint of its theme of underlying mathematical understandings, in which consideration is given to the historical development of number concepts and notational systems (eventually extending to duodecimal and binary scales of notation, particularly); to the fundamental operations and related laws applied to whole numbers and to common and decimal fractions (including per cents); to selected aspects of the theory of natural numbers; to the extension of number concepts to embrace negative numbers, rational and irrational numbers, and real numbers; and to work with measures and approximate data.

A more careful examination of Swain's book reveals numerous variations from more or less similar volumes in terms of the scope of content covered and its organization and treatment. Without doubt the most significant difference in this connection is the extent to which Swain interprets numerous aspects of arithmetic in terms of ideas from "modern mathematics," particularly the concept of *set*. This will be the most influential single factor in determining a reader's reaction to *Understanding Arithmetic*. Therefore, the present review will be devoted chiefly to Swain's development and application of the *set* concept and related ideas.

The notion of a set is introduced relatively early in the book (Chapter 2, "Sets and Numbers"). Some of the unique symbolism or notation associated with sets and subsets is used to advantage. *Number* is more or less re-defined in terms of the set concept, following an earlier distinction between *number* and *numeral*.

In this connection, Swain admittedly does not hold consistently to the essential difference between these terms in the way in which he uses them: "This important distinction between 'number' and 'numeral' sometimes escapes

notice, because the term 'number' is widely used to refer to either concept. In this book we shall make the distinction whenever it seems wise to do so, but at other times we shall follow the common practice." (p. 4)

Subsequent sections of the book are developed from set concepts to varying degrees, with greatest use of these ideas being made in Chapter 3 ("Number Operations"). Here the principal operations of addition and multiplication, and the inverse operations of subtraction and division, are treated from a set standpoint. For example, the sum of two natural numbers is defined in relation to the union of disjoint sets. Similarly, the product of two natural numbers first is defined in relation to the union of equivalent disjoint sets, and then is re-defined in set terms to make multiplication an operation "in its own right" rather than a special or extended form of addition. The interpretations of the inverse operations of subtraction and division are based chiefly upon the idea of subsets and upon the principal operations of addition and multiplication.

In dealing with the union of two disjoint sets, e.g., X and Y, Swain relies exclusively upon the symbol "$X+Y$" and does not introduce and use the more unique symbol "$X \cup Y$" to denote this idea of union. This is but one of numerous instances in which Swain admittedly deals with certain aspects of set theory without being highly rigorous, either conceptually or notationally or both. In some instances this lack of rigor is acknowledged explicitly, as in connection with the author's definition and discussion of *real numbers* on p. 187. In other instances there is no explicit acknowledgement of a lack of rigor that is quite intentional on the author's part and that is justified in his eyes in light of the mathematical background and needs of the main audience he is addressing.

In the early work with sets and their application to number operations, set concepts also are used as the basis for developing the commutative, associative, and distributive laws. Furthermore, set concepts form the basis for distinguishing between two interpretations of the 0 symbol. In reference to the system of whole numbers (designated by the numerals 0, 1, 2, 3, 4, · · ·), 0 is interpreted in relation to the idea of an *empty* set. Then, in reference to the system of integers (designated by the symbols · · · , −3, −2, −1, 0, 1, 2, 3, · · ·), 0 is interpreted in relation to the idea of a *null* set.

Swain's subsequent treatment of computational procedures for performing the fundamental operations within the framework of our numeral system (Chapters 4 and 5) is disappointing to this reviewer. For one thing, the chapters omit many logical sequential steps in the development of ultimate computational procedures and thus take too much for granted in relation to the student's own mathematical background. For another thing, the author's discussion of methods of compound subtraction is both inadequate and misleading. This is particularly true of his interpretation of the vast body of research on the relative merits of the *decomposition* and *equal additions* procedures—which Swain elects to identify by expressions other than these two that are so commonly accepted and used in our professional literature. His subsequent discussion of methods of estimating quotient digits when dividing by two-digit divisors is much more satisfactory. Here again, however, he elects to use expressions (*one-step rule* and *two-step rule*) different from those commonly accepted and used in the professional literature (*one-rule method* and *two-rule method*, or *apparent method* and *increase-by-one method*).

To return to applications of the set concept and related ideas, these are expanded in a limited way to embrace "Fractions" in Chapter 8. Here a fraction is characterized first in terms of a relative comparison between two sets, i.e., as a ratio. This leads to a subsequent interpretation of a fraction as a quotient and hence as a number *per se*. However, the definition of a fraction is not extended to a higher and more abstract level. That is, Swain does not develop and use the concept and unique notation of a fraction as an element in the set of all ordered pairs of integers $(a, b;$ where $b \neq 0)$, which in turn leads to a "set definition" of a rational number.

Swain uses the set concept and related ideas, however, as a basis for his discussion of "Reducing Fractions" and "Adding Fractions" in particular. In the latter instance, for example, by treating two fractions as proper subsets of an appropriate "base set" (or multiple thereof), the addition of these fractions is interpreted in terms of the union of two disjoint sets. In general, however, ideas associated with "modern mathematics" have been used neither as extensively nor as explicitly in the treatment of fractions as in the treatment of integers.

Swain's handling of computational procedures in his chapters on "Fractions" (Chapter 8) and "Decimals" (Chapter 9) is as disappointing to the present reviewer as was his treatment of such techniques in Chapters 4 and 5, mentioned previously. The same major criticism applies—many logical sequential steps in the development of ultimate computational procedures are omitted, thus taking too much for granted in relation to the student's own mathematical background. Also, in Chapter 9, the discussion of techniques for placing a decimal point in a quotient is considered inadequate.

On the other hand, this reviewer is impressed favorably with numerous aspects of Swain's material dealing with duodecimal and binary scales of notation (Chapter 6), with the theory of natural numbers (Chapter 7), and with work relating to measures and approximate data (Chapters 11 and 12). His inclusion of selected theorems, etc., and their proofs is a particularly strong feature.

In summary, we may expect mixed reactions to *Understanding Arithmetic*, particularly from the standpoint of Swain's relative emphasis upon the concept of set. At the one extreme there will be persons who feel that none of this "modern stuff" should have been included. At

the other extreme there undoubtedly will be persons who feel that Swain has been much too superficial in his treatment of ideas from "modern mathematics"; that he has compromised entirely too much with mathematical rigor.

As far as this reviewer is concerned, either extreme position would be unfortunate and unwarranted. The most reasonable opinion would seem to lie somewhere between these extremes. In relation to the readers for whom *Understanding Arithmetic* is intended, Swain's handling of the set concept and related ideas from "modern mathematics" should be more helpful than not. His efforts in this connection should be welcome and are deserving of commendation rather than condemnation.

Some things have been mentioned as shortcomings in the eyes of the present reviewer. To other persons, Swain's treatment of the topics in question might be adequate for his intended purposes. Such a difference in reaction would be quite understandable, since for the most part these are matters of opinion rather than fact.

In any event, *Understanding Arithmetic* should represent a worthwhile addition to our professional literature for persons seeking a better understanding of arithmetic content *per se*.—*J. Fred Weaver, Boston University, Boston, Massachusetts.*

INSTRUCTIONAL MATERIALS

Direct-O-Percenter. Educational Supply and Specialty Company, 2823 Gage Avenue, Huntington Park, California; or A. C.. Vroman, 367 S. Pasadena Avenue, Pasadena, California. 50 cents each, less 10 per cent in lots of 10 or more.

This cardboard device for use with percentage problems consists of a right triangle printed on cardboard (11″×14″) with the bottom leg divided into one hundred equal parts to serve as a per cent scale. A duplicate of this scale appears on a separate cardboard strip which can be placed on top of the triangular figure at the appropriate place to solve a proportion. A black thread attached at the upper vertex of the right triangle is used to form another right triangle for the graphic solution of the given proportion. Directions for using the device are printed on the cardboard also.

This device will be found to be more useful in connection with similar triangles in geometry than for teaching the solution of problems involving per cent. The reason for this is that although the meaning of per cent is geometrically shown, the device does not show how to solve the proportion. That is, the device solves the proportion but does not indicate how the student can compute with number symbols to solve the proportion. Another drawback of the device is that no per cent larger than 100 can be used. Some students will continue to think that it is impossible to have more than 100 per cent. —*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

Learning New Numbers—Fractions. Filmstrip House, 347 Madison Avenue, New York 17, New York. Complete set, $40.00; each filmstrip, $6.00.

The script for this set of eight color filmstrips was written by Jean Kenner and Morton Kenner, and the artwork was done by Clyde Strohsahl and Woodrow Jackson. The titles of the filmstrips are: 1. "What is a Fraction?" 2. "Working with Equal Fractions and Reducing Fractions," 3. "Working with Like Fractions and Improper Fractions," 4. "Adding with Fractions," 5. "Subtracting with Fractions," 6. "Beginning to Multiply and Divide with Fractions," 7. "Multiplying with Fractions," and 8. "Dividing with Fractions." Throughout the series, use is made of a boy, Alan, and his imaginary pet, Gerry-Graph.

This set is attractive to view and easy to read. The whole set or parts of it could be used at the junior high school level for remedial purposes, even though the set is primarily intended for grades five and six.

In the opinion of the reviewer, some parts of the filmstrips are weak from a mathematical standpoint. The meaning of a fraction given in Filmstrip 1 emphasizes that a fraction tells "what part" of something we have, rather than emphasizing that a fraction tells "how much" we have. This subtle difference seems important to the reviewer. The result is that children will miss the most important idea about fractions: namely, that $\frac{1}{4}$″ merely means three of the new unit of measure, fourth-inch. This idea of a new unit of measure is not specifically mentioned in the set, and its omission prevents the rest of the set from giving fuller meanings behind computation with fractions.—*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

"... there is literally nothing taught in the traditional curriculum of secondary school mathematics that could not have been taught in the seventeenth century."—*A. E. Meder, Jr., Rutgers University, New Brunswick, New Jersey.*

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

High school examinations in Scandinavia

Editor's Note: We are deeply indebted to Dr. Howard F. Fehr for his generosity in making available the two Scandinavian examinations printed in this month's column. The reader undoubtedly knows Professor Fehr as President of the National Council of Teachers of Mathematics and Head of the Department of the Teaching of Mathematics at Teachers College, Columbia University.

Most of the Scandinavian countries (Denmark, Finland, Iceland, Norway, and Sweden) are reputed to have attained a high standard of living and a high level of education and technology. Illiteracy is virtually non-existent. What kind of mathematical training do secondary students in Scandinavia receive?

Tests seem to supply quick clues to such a question. In this month's column, two Scandinavian examinations have been reprinted for your consideration. You will undoubtedly find most questions of interest; certainly they should prove a challenge to your senior students.

For perspective, it seems helpful to note that Scandinavian secondary schools are highly selective in character. For example, in Norway, after a common and tuition-free elementary schooling for all until age 14, only slightly more than half of high school age children are selected to continue their education. Of these, a small percentage attend the *realskole*, where they receive a general education. A sternly academic training is offered the remainder of these selected students in the *gymnasium*.

All *gymnasium* students study such subjects as English, German, and French in addition to the native tongue; history, citizenship, and geography; mathematics and biology; art and gymnastics. Each student also undertakes one of five areas of specialization: Latin, English, mathematics and physics, Old Norse, or natural science. After five years of such training, the student, now 19 or 20, is ready to graduate. To do so, he must take the *examen artium* prepared by the national Council of Secondary Education. This examination also serves as a university matriculation test.

The following test is the examen artium of 1956 for students majoring in mathematics and science. The student is given three hours to work on it.

Norway leaving examination, 1956[1]

(Examen Artium)

I

1. Write the relationships between the roots and the coefficients in the equation $x^3+ax^2+bx+c=0$.
2. Use these relationships to prove that if the three roots of the equation are in geometric sequence, then one of the roots always equals $-\sqrt[3]{c}$, and this same root also equals $-b/a$.
3. As an example let c have the value -64. What, then is the value of the root in part 2? Give the other two roots expressed in terms of a, and find the values of a for which these two roots will be real.

[1] Reported in *Nordisk Mathematisk Tidskrift* (Oslo, 1956), IV, Part 3, 172–173. Translated by Kay Piene, Rektor, Institute of Pedagogy, Oslo.

II

In a hemisphere of radius r, there is inscribed a triangular pyramid $ABCT$. The base of the pyramid is an isosceles triangle ABC with base BC which is inscribed in the circular base of the hemisphere. The vertex T of the pyramid is on the line perpendicular to the base of the pyramid at the mid-point of BC.

1. Let the distance from A to BC be x. Compute the volume of the pyramid in terms of r and x.
2. If r is constant, the pyramid has a maximum volume for a certain value of x. Find the maximum value of the volume and the corresponding value of x. Show that value of the volume is really a maximum value.
3. If $x=3/2\ r$, two planes are passed through the radius of the hemisphere perpendicular to its base, one plane containing B, the other containing C. Another plane is passed through BCT. Compute the area in terms of r of the curved surface of the hemisphere bounded by these three planes.

III

1. Given the ellipse $4x^2+9y^2=36$ and the straight line $x=3$. A point P on the straight line has the ordinate q. What is the equation of the polar of P with respect to the ellipse? Show that the polar always passes through a fixed point A, as P moves on the straight line, and give the coordinates of point A.
2. The point M with coordinates $(m,0)$ is also given. Find the equation of the locus of the intersection point of the polar of P and the straight line through points M and P, as P moves on the straight line $x=3$. The curves of the loci for different values of m all pass through A and M. Explain how you can tell in advance that this must be the case.
3. The locus of part 2 has the equation

$$4x^2+3(3-m)y^2-4(m+3)x+12m=0.$$

Describe the conics this equation represents for various finite values of m. (Give only the type of conic without finding the center, axis, length of axis, etc.) What is the geometric interpretation of the equation if m is placed equal to 3?

The following is a competitive examination open to all students in Denmark, Finland, Iceland, Norway, and Sweden. This is a "take-home" test, much in the same spirit as that suggested by Mrs. Elder in the previous month's Testing Time column! The Scandinavian student signs and encloses with his replies a declaration that his solutions were made without assistance from other persons.

INTER-SCANDINAVIAN COMPETITION IN MATHEMATICS FOR HIGH SCHOOL STUDENTS, 1957[2]

1. P is a point on an ellipse and Q is the projection of the center of the ellipse on the tangent to it at P. Show that the maximum value of the distance PQ is equal to half the difference between the major and minor axes of the ellipse.
2. Given a triangle ABC and an arbitrary point P in space. Prove that

$$3(PA^2+PB^2+PC^2)\geqq AB^2+BC^2+CA^2.$$

What is the position of P for which the equality sign is valid?

3. Given a triangle ABC. On the side BC a point A_1 is situated, and similarly points B_1 on AC and C_1 on AB so that

$$\frac{BA_1}{A_1C}=\alpha;\quad \frac{CB_1}{B_1A}=\beta;\quad \frac{AC_1}{C_1B}=\gamma.$$

The lines AA_1, BB_1, and CC_1 form a smaller triangle in the interior of the given triangle. Compute the relation

[2] Reported in *Nordisk Mathematisk Tidskrift* (Oslo, 1956), IV, 166. Translated by Kay Piene, Rektor, Institute of Pedagogy, Oslo, and edited by Dr. Howard F. Fehr.

between the areas of the smaller triangle and the given triangle expressed in terms of α, β, and γ. The result should be simplified as much as possible.

4. Show that for the function defined by:

$$y = \frac{1}{\sin x} - \frac{1}{x},$$

y increases as x increases in the interval $0 < x \leq \pi/2$. Compute $\lim y$ as $x \to 0$.

5. A sequence of numbers are defined as follows:

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 = 5, \cdots$$
$$a_{n+2} = a_{n+1} + a_n \cdots.$$

Designate the sum of the first n terms in the sequence by S_n.

a) Compute S_n expressed by one of the members of the sequence.
b) Show that $a_n \cdot a_{n+2} - a_{n+1}^2 = (-1)^{n+1}$.
c) Find $\lim (a_{n+1})/a_n$ as $n \to \infty$.

6. Given a convex polygon with n sides and having the following properties:
a) Two diagonals are never parallel.
b) Three diagonals never pass through the same point other than the vertex of the polygon.

How many intersection points of the diagonals can be found respectively inside, and outside the polygon (not on the polygon)?

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING
April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA
June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

EIGHTEENTH SUMMER MEETING
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Originality in geometry

By William E. Beeman, Arlington State College, Arlington, Texas

While teaching seventh and eighth grade mathematics in the North Texas Laboratory School, Denton, Texas, I had a most pleasant experience teaching a unit on plane geometry that I would like to share with other teachers.

It has always been my practice to read the morning newspaper before going to school. One morning my favorite comic strip detective, Kerry Drake, and his assistant Bryson were using fragments of a headlight lens pieced together to construct a circle and, consequently, identify the automobile from which it came. I wondered if any of my students were noticing this interesting use of geometric constructions—they all like "funny papers."

Other work was planned for that day, but I decided to "spring" this comic-strip episode on the classes and see what happened. Naturally, the students were interested in the fact that such a detection job could be done by using geometry. In particular, they wanted to know how a circle could be constructed if one had only a part of it with which to start. We could do nothing else but start discussing the various fundamental definitions and constructions necessary to construct this circle.

We had to learn how to use a ruler and compass by drawing line segments and angles of various sizes. From these drawings the ideas of bisect and perpendicular were intuitively understood as related to a line segment and an angle.

Then each student was put on his own with all the time he wanted, and with a problem. Of course, some pupils did not get very far but they had much fun trying, and that is what some had never done much of before.

The problems were given in this order.

1. Construct the bisector of an angle.
2. Bisect a line segment.
3. Construct a perpendicular to a line from a point on the line and also from a point not on the line.
4. Construct the perpendicular bisector of a line segment.
5. Construct a circle containing three points not on a line.

It was this last problem that was the first objective of our work, but after it had been successfully completed most of the students wanted more. As interest was so high, progress was rapid. More and more geometric questions arose until finally I was kept busy finding or "thinking up" new problems, or checking original efforts at solutions.

The following original constructions show the originality and rapid progress I have mentioned.

The proofs are my own. The students could only make measurements and use intuitive reasoning.

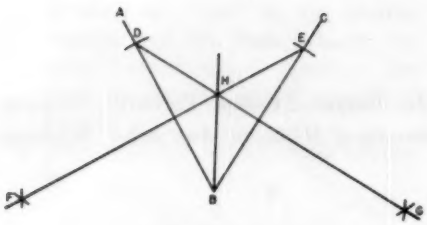

Figure 1

This is an original construction by Anne McGuffin, seventh grade, North Texas Laboratory School, Denton, Texas.

Problem I:

To bisect a given angle ABC.

Construction:

1. From the vertex B mark off equal distances BD and BE on the sides of the given angle ABC.
2. With B and E as centers and with any given radius, draw arcs intersecting at point G. With the same radius and with D and B as centers, draw arcs intersecting at point F.
3. Draw lines DG and EF. Connect their point of intersection H with the vertex B.
4. The resulting line BH bisects the given angle.

Proof:

On the above figure draw line segments DE, BF, BG, DF, and EG.

1. $BD = BE$
   Construction.
2. $BF = BG$
   Construction.
3. $DF = EG$
   Construction.
4. $\triangle BDF$ is congruent to $\triangle BEG$.
   Three sides of one triangle equal to the corresponding three sides of the other triangle.
5. $\angle BDF = \angle BEG$
   Corresponding angles of congruent triangles are equal.
6. $\triangle DBE$ is an isosceles triangle.
   Definition.
7. $\angle BDE = \angle BED$
   Equal angles of an isosceles triangle.
8. $\angle FDE = \angle GED$
   Equal angles added to equal angles result in equal angles.
9. $DE = DE$
   Same line segment.
10. $\triangle FDE$ is congruent to $\triangle GED$.
    Two sides and included angle of one triangle equal to corresponding two sides and included angle of the other triangle.
11. $\angle DEF = \angle EDG$
    Corresponding angles of congruent triangles.
12. $DH = EH$
    Sides opposite equal angles of triangle DEH.
13. $BH = BH$
    Same line segment.
14. $\triangle BDH$ is congruent to $\triangle BEH$.
    Three sides of one triangle equal to corresponding three sides of the other triangle.
15. $\angle DBH = \angle EBH$
    Corresponding angles of congruent triangles.
16. BH bisects an given angle ABC.
    Definition.

Problem II:

To construct a line perpendicular to a given line AB from an external point C.

Construction

1. Using A as center draw arc DE with radius AC.
2. Using B as center draw arc FG with radius BC.
3. Connect C with the intersection of arcs DE and FG. The resulting line CH is perpendicular to AB or to AB extended.

Proof:

On the construction described above draw AC, AH, BC, and BH. (Figure 2a)

1. $AC = AH$
   Construction.

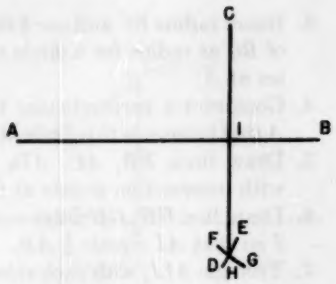

Figure 2a

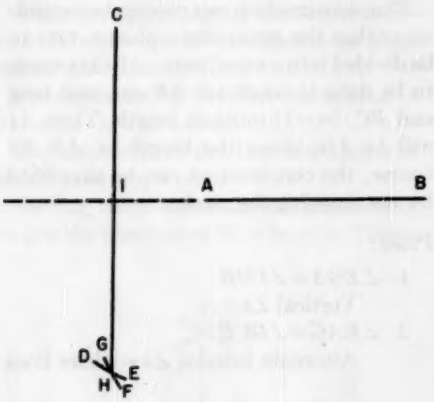

Figure 2b

This is an original construction by Bill Farmer, seventh grade, North Texas Laboratory School, Denton, Texas.

2. $BC = BH$
   Construction.
3. $AB = AB$
   Same line segment.
4. $\triangle ACB$ is congruent to $\triangle AHB$.
   Three sides of one triangle equal to corresponding three sides of the other triangle.
5. $\angle CAB = \angle HAB$
   Corresponding angles of congruent triangles.
6. $AI = AI$
   Same line segment.
7. $\triangle ACI$ is congruent to $\triangle AHI$.
   Two sides and included angle of one triangle equal to corresponding two sides and included angle of the other triangle.
8. $\angle AIC = \angle AIH$
   Corresponding angles of congruent triangles.
9. $\angle AIC + \angle AIH = 180°$
   CIH is a straight angle and definition of straight angle.
10. $2\angle AIC = 180°$
    Substitution axiom.
11. $\angle AIC = 90°$
    Division axiom.
12. CH is perpendicular to AB.
    Definition.

Similarly the proof for Figure 2b can be shown.

Problem III:

To construct a right triangle that has a perimeter equal to the length of a given line segment AB. (Figure 3)

Construction:

1. Using any radius less than $\frac{1}{2}$ the length of given line segment AB, draw a circle with its center O on the given line segment and passing through the point B.
2. Construct the perpendicular to AB at O intersecting the circle at some point C.
3. With A and C as centers and using any given radius, draw intersecting arcs at D and E.
4. Draw DE intersecting AB at F.
5. Draw FC.
6. The triangle COF is one of the many triangles which fit the given conditions.

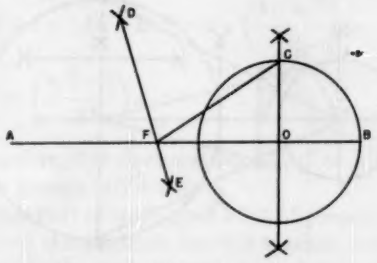

Figure 3

This is an original construction by Jack Baker, eighth grade, North Texas Laboratory School, Denton, Texas.

The truth of this construction is rather obvious if one will think of DE as being the perpendicular bisector of a line segment drawn from A to C; then $AF=FC$. There is, of course, no unique triangle FOC. But one must be sure to have OB less than one half of AB.

Proof:

1. $AF=FC$
   Any point (F) on $\perp$ bisector of a line segment (of AC) is equidistant from its end points.
2. $CO=OB$
   Radii of same $\odot$ =.
3. $AB=AF+FO+OB$
   Whole = sum of its parts.
4. $AB=FC+FO+OC$
   Substitution axiom.

Problem IV:

Construct an equilateral triangle that has a perimeter equal to the length of a given line segment AB.

Construction:

The student immediately saw that his problem was to trisect the given line segment AB. Once this was done the rest was to him trivial. He did this as follows.

1. Bisect line segment AB and use $\frac{1}{2}$ the length of AB as a radius to draw a circle with center B.
2. Construct a perpendicular to AB at B that intersects the circle at points C and D.

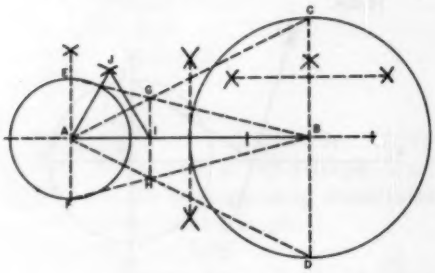

Figure 4

This original construction was done by Lynn Wilson, eighth grade, North Texas Laboratory School, Denton, Texas.

3. Bisect radius BC and use $\frac{1}{2}$ the length of BC as radius for a circle with center at A.
4. Construct a perpendicular to AB at A that intersects this circle at E and F.
5. Draw lines EB, AC, AD, and FB with intersection points at G and H.
6. Draw line GH. GH intersects AB at I so that AI equals $\frac{1}{3}$ AB.
7. Triangle AIJ, with each side equal to AI, is the required triangle.

This construction can readily be extended so that the given line segment AB can be divided into n equal parts. All that needs to be done is construct AE one unit long and BC, $(n-1)$ units in length. Then AI will be $1/n$ times the length of AB. Of course, the construction can be simplified by not drawing the circles.

Proof:

1. $\angle EGA=\angle CGB$
   Vertical $\angle s$ =.
2. $\angle EAG=\angle GCB$
   Alternate interior $\angle s$ = where lines $\parallel$.
3. $\triangle AEG\sim\triangle GCB$
   $2\angle s$ =.
4. $EA:CB=EG:GB$
   Corresponding sides are proportional.
5. $EA:CB=1:2$
   Construction.
6. $1:2=EG:GB$
   Substitution axiom.
7. $GI\parallel EA$
   Lines $\perp$ same line are $\parallel$. (Reader is left to supply intermediate steps leading to this reason.)
8. Hence $EG:GB=AI:IB$.
   Line $\parallel$ one side $\triangle$ and intersecting other two divides them proportionally.
9. $1:2=\mathrm{AI:IB}$
   Substitution axiom.

Proof:

The following proof is for the more general case which leads to the division of a line segment AB into n equal parts.

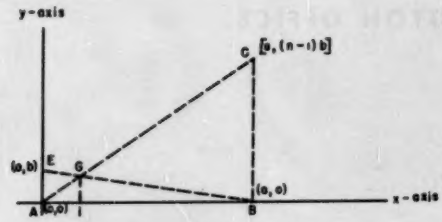

Figure 5

Take the upper part of the construction and use rectangular coordinates as illustrated in Figure 5.

The equation of the line containing A and C is $ay=(n-1)bx$.

The equation of the line containing E and B is $bx+ay=ab$.

Solving these equations simultaneously we get the abscissa of G to be a/n. Therefore, since the length of AB is a, $AI=1/n$ (AB).

Several other attempts were made to trisect a given line segment. The construction in Figure 6 was one of the better misses.

Problem V:

Trisect a given line segment AB. (Figure 6)

Attempted Construction:

1. Bisect AB at E making sure AB equals AC equals BC.
2. Draw line CD intersecting AB at E.
3. Bisect CE to determine point F.
4. Use radius AF and centers A and B to draw arcs that intersect AB at G and H.

The student was hoping that AG, GH, and HB were all equal, but, of course, it can be shown that this is not true.

Proof:

Here we are to prove that AG, GH, and HB are not all equal. On the above construction, draw AF and BF. Let the length of AB be a, then $CE=a\sqrt{3}/2$ and $EF=a\sqrt{3}/4$. Also, $AF=AH=a\sqrt{7}/4$. Likewise $BF=BG=a\sqrt{7}/4$. It follows that,

$$AG=\left(\frac{4-\sqrt{7}}{4}\right)a$$

$$BH=\left(\frac{4-\sqrt{7}}{4}\right)a$$

$$GH=a-(AG+BH)=a-\left(\frac{4-\sqrt{7}}{2}\right)a$$

$$=\left(\frac{\sqrt{7}-2}{2}\right)a.$$

Therefore, GH does not equal AG or HB even though $AG=HB$.

This unit of work gave a lot of pleasure to both the students and the teacher, and certainly the students gained much knowledge and enthusiasm that will help in other mathematics courses.

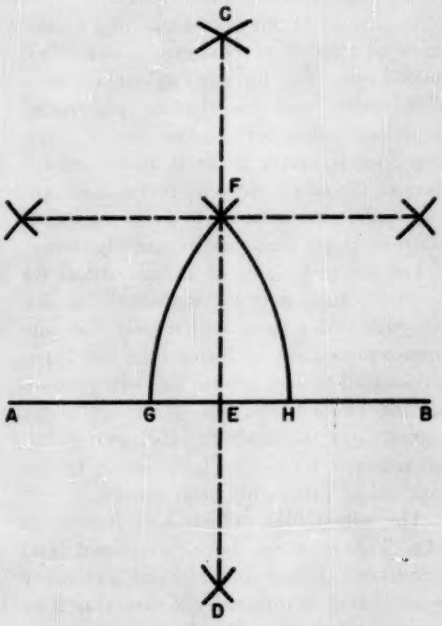

Figure 6

This original construction was done by Bill Farmer, seventh grade, North Texas Laboratory School, Denton, Texas.

New business procedures in the Washington office

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

Every organization should be constantly on the alert to find ways to increase the accuracy and efficiency with which it renders its services and discharges its duties. We have tried to make this belief both our philosophy and our working plan in the headquarters office of the Council.

As the Council has grown in size and activity, the office too has expanded. New procedures, new business forms, and new files have been needed. The systems that we have used have been largely of our own invention and have grown out of the particular needs of the office.

Recently we have felt that the entire office system should be rethought. In our type of work, with many thousands of orders to process each year, some errors are inevitable. We could reduce the number of errors by adding more elaborate procedures including checks and double checks. But our budget could not support the extra clerical help needed for this approach. We needed to improve accuracy without increasing labor.

We began to wonder whether we might not have grown large enough to be able to use some of the systems used by the large commercial companies engaged in our type of work. Accordingly, during the past summer, the executive secretary and some members of the staff visited the order-processing departments of two large textbook publishers and the subscription department of a magazine of large circulation.

We were both pleased and surprised to find that the systems that had grown out of our own experience coincided rather closely with those used by these large companies. We had been "on the right track." We were also interested to learn that their problems of achieving accuracy and efficiency were similar to ours.

As a result of our investigation we have made a number of changes in our office procedures. We have designed for our membership and subscription processing some new office forms. The use of these new forms should make it more certain that all the many steps involved in changing a subscription order to a correctly filed addressograph plate are accurately done.

For the processing of charge orders we have designed a new "snap-out" invoice set. Our order clerk will merely roll one preassembled set of forms into her typewriter and in one typing she will prepare all the needed invoices, statements, file copies, and the mailing label. Some improvements have also been made in our method of processing cash orders.

The office files, which had grown up like Topsy, have been simplified and integrated. Filing and locating necessary records and correspondence should now be easier and faster. Our bookkeeping has been simplified, so that we are keeping necessary financial records with less work.

We are hoping that these changes will result in improved service to you, our customers, during the coming months.

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Henry Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

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